3 1761 08091839 4

UNIV. OF
TORONTO
LIBRARY

Math

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. L.

141781
17/12/18

LONDON:
CHARLES AND EDWIN LAYTON,
FARRINGDON STREET, E.C. 4.

PARIS: 8, RUE LAMARTINE, 8. MELBOURNE: McCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

1917.

[*The Council of the Institute of Actuaries wish it to be understood that while they consider it their duty to give, from time to time, publicity to certain of the papers presented to the Institute, and to abstracts of the discussions at the Sessional Meetings, they are not responsible for the opinions put forward therein.*]

[ENTERED AT STATIONERS' HALL.]

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FARRINGDON STREET, E.C. 4.

CONTENTS OF VOL. L.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

The Valuation of Double Endowment Assurances. *By* Hubert Vaughan, F.I.A., *of The Mutual Life and Citizens' Assurance Company, Limited.*

1. The papers of Mr. Lidstone on the valuation of Endowment Assurances (*J.I.A.*, vol. xxxiv, p. 61, and xxxviii, p. 1) are well-known, and Mr. Ackland has extended the investigation to the case of Joint Endowment Assurances (*J.I.A.*, vol. xxxviii, p. 61). The following investigation into the applicability of a similar method to Double Endowment Assurances may, therefore, be of interest. Mr. Lidstone's investigation is so complete and has been so fully tested in practice that the simplest method of investigation has been to compare Double Endowment functions with those for Endowment Assurances, and draw conclusions from the established properties of the latter.

2. The error when a Double Endowment business is valued by Mr. Lidstone's method (taking $r=c$) can be readily estimated in particular cases from Table A appended. This table displays, for the Text-Book Table at 3 per-cent interest, the first four terms when the function $\frac{1}{2}(A_{x\overline{n}|}+A_{x\overline{n}|}^{\;\;1})$ is expanded in powers of c^x,* and, for comparison, Table C gives the corresponding particulars for $A_{x\overline{n}|}$. Table B, showing the terms in a similar expansion of $a_{x\overline{n}|}$, is quoted for the sake of comparison with the figures for the $O^{M(5)}$ Table (*J.I.A.*, vol. xxxviii, p. 5). Owing to the limitation in the number of decimal places retained in calculating these tables the last figure will not be invariably correct, but the degree of accuracy is sufficient for

* The method of developing such a series and the use of the table are set out in *J.I.A.*, vol. xxxviii, p. 3.

our purpose. The difference between columns 5 and 6 at ages 15 and 20 is due to the fact that the Makeham constants do not express the Text Book Table below age 30.

3. It will be seen that the Double Endowment series converges rapidly, but not so rapidly as that for Endowment Assurances. It will also be seen that the terms after the first are opposite in sign to the corresponding terms in the series for $A_{x\overline{n}|}$ and the same as those for $a_{x\overline{n}|}$. It follows that the error in the value of Sum Assured when r is taken $= c$ would be in the direction of understatement, as would that in the value of Premiums. In an Endowment Assurance valuation on the same basis there is a tendency to overvalue the Sum Assured and undervalue the Premiums, the error being cumulative as regards the Reserve. In a Double Endowment valuation both items would tend to be undervalued, the errors counteracting; and, in consequence, though we should expect the error in valuing the Sum Assured to be greater than in the former case, the error in the Reserve need not be excessive. However, as we will see later, the final error is likely to be in the direction of understatement, the "wrong direction", and so the Endowment Assurance method requires modification when applied to this class of policy.

4. In a Lidstone valuation with $r=c$, conditions unfavourable to accuracy are generally speaking:

1. A high mean maturing age.
2. A wide range of maturing ages.

Fortunately these conditions do not in normal cases co-exist. For example, an office that habitually canvassed for policies at maturing ages ranging from 30 to 60 would naturally have a wide range in most groups, much wider than an office that published prospectus rates only for ages 50, 55 and 60, but the mean ages in the former case would be lower and so the Lidstone error perhaps not greater. It has been pointed out (*J.I.A.*, vol. xxxviii. p. 47) that the same principle of compensation runs through the valuation groups in any particular business. If we take policies maturing 40 years hence the average maturing age will be high, but the range cannot be wide, say from 60 to 75. On the other hand policies maturing 10 years hence might range between maturing ages as wide apart as 30 and 75, but in such a case the mean age for the group would be lower.

5. If we assume that the mean maturing age will not

exceed 56, it is not likely that the difference between mean ages based on the ratio c and c^2 will exceed, say, 2·3 years. The error will then not exceed ·5 of the term involving c^{2x} in the expansions appended and so the maximum error in any group may be estimated as follows for the functions stated:

n	$\frac{1}{2}(A_{x\overline{n}\rvert} + A_{x\overline{n}\rvert}^{\ \ 1})$	a_{xn}	$A_{x\overline{n}\rvert}$
11	·0010	·006	·0002
21	·0011	·012	·0003
31	·0010	·012	·0003

6. On the alternative assumption that the mean age is as high as 61, the difference between mean ages based on c and c^2 would hardly exceed 1·5 years. The error would be in that case less than, say, ·3 of the term involving c^{2x}, the limits for the different functions being as follows:

n	$\frac{1}{2}(A_{x\overline{n}\rvert} + A_{x\overline{n}\rvert}^{\ \ 1})$	a_{xn}	$A_{x\overline{n}\rvert}$
11	·0014	·008	·0002
21	·0017	·017	·0005
31	·0014	·017	·0005

7. Both these tables are on the stringent side. Errors so large would usually occur only in a few groups, and it may be expected in practice that the error in the valuation of Sum Assured over the whole table will not exceed ·001 of the endowment benefit.

8. The valuation multiplier for the Premiums is of course the same as in the case of Endowment Assurances, and most of the remarks in *J.I.A.*, vol. xxxviii, p. 16 *et seq.*, apply. The only point that calls for mention is that the Net Premium under a Double Endowment table does not increase with the age when the term is constant, and so the mean age based on the distribution of the Net Premiums would be slightly lower than that of an Endowment Assurance table with the same age distribution of the Sum Assured. As a result, if the mean age is based on Sum Assured, there is a slightly wider safety margin in the value of the Premiums in the case of the Double Endowments.

9. We have now to consider the effect of departures from

Makeham's law. Irregularities due to lack of smoothness in the graduation of such tables as the H^M may be ignored, but the effect of definite departures from Makeham's law over sections of the table has to be considered. In the case of the Text-Book table a reference to Table B will show that in the case of very young assured lives the value of the Premiums will be materially understated. The numerical effect on a valuation will depend on the proportion of such young lives to the total, and since Endowment Assurance policies are sometimes issued at entry ages as low as 11 it may be as well to call attention to it. The resulting error in the value of Premiums is in the safe direction and perhaps can be regarded as an addition to the margin of safety.

10. It will be seen from Table A that the effect of this divergence from Makeham's law in the Text-Book Table does not affect the value of Sum Assured under Double Endowments as much as under Endowment Assurances because the errors are in opposite directions for different future terms, and accordingly no especial danger need be apprehended from this quarter in a Double Endowment valuation.

11. If there is a Bonus payable in the same way as the Sum Assured, namely, the full amount at maturity and a half on previous death, there is no reason why any large error should be expected from using the same valuation multiplier for the Bonus as for the Sum Assured, unless the Bonus is proportionately large in amount and unusual in distribution.

12. It appears then that if we value a Double Endowment class by Mr. Lidstone's method, putting $r=c$, and add to the result an adjustment equal to ·001 of the endowment benefit, the result will be close enough for practical purposes, and the direction of error will be practically always on the safe side. The result of an actual valuation is quoted later in confirmation.

13. Another method that suggests itself is to find, by trial, a value of r that will represent as closely as possible the rate of increase in the differences of $\frac{1}{2}(A_{x\overline{n}|}+A_{x\overline{n}|}^{\;\;1})$. Of course the difficulty arises that the value of r appropriate to this function will not be that best applicable to the annuity, a point that does not arise in an Endowment Assurance valuation because the differences of $A_{x\overline{n}|}$ and $a_{x\overline{n}|}$ are in proportion. A very accurate valuation could be made by finding separate mean ages for Sum Assured and Premium based on slightly different values of r, but the gain in exactness would not repay the inconvenience

of keeping two sets of Z sheets and having two subsidiary valuation factors instead of one.

14. If p is the value of r that would reproduce the result of an individual valuation of Sum Assured and q the corresponding value for the Premiums; then p will be slightly less than q. Now if we use a value between p and q for both Sum Assured and Premiums the result will be to undervalue the Sum Assured and overvalue the Premiums, both errors being on the wrong side as regards the Reserve. If we take a ratio higher than q we will undervalue both items, if a ratio lower than p we will overvalue both; so it will be interesting to see if a value of r can be found at which the errors in valuing Sum Assured and Premiums will cancel.

15. The following table shows values for r for the function $\frac{1}{2}(A_{xn|} + A_{x\overset{1}{n}|})$ determined by the formula

$$r = \left(\frac{u_M - u_{M+t}}{u_{M-t} - u_M}\right)^{\frac{1}{t}}$$

for the Text-Book Table at 3 per-cent. interest.

n	Taking M as 55 and t as			Taking M as 60 and t as	
	15	10	5	10	5
10	1·082	1·084	1·084	1·078	1·080
20	1·074	1·078	1·078	1·068	1·071
30	...	...	1·083	1·063	1·065

Corresponding values for $\text{a}_{xn|}$ are:

n	Taking M as 55 and t as			Taking M as 60 and t as	
	15	10	5	10	5
10	1·089	1·090	1·092	1·087	1·087
20	1·080	1·087	1·088	1·084	1·084
30	...	...	1·046	1·074	1·084

The presence of so low a value as 1·046 at future duration 30 may be attributed to the change in the law of the table below age 30, and considering only values that do not involve ages below 25 the figures lead to the conclusion that p is about ·01 lower than q. Adopting Mr. Lidstone's estimate of 1·085 for q,

p would be, say, 1·075. If the maturing ages are rather lower the values might be say, 1·09 and 1·08 respectively. Taking the safer value in each case we can put $p=1{\cdot}075$ and $q=1{\cdot}09$.

16. Now to estimate the point at which errors in valuing Sum Assured and Net Premiums will balance it is necessary to consider the relative effect on each of an alteration in r. The alteration of $x+n$ from 55 to 56 makes the following differences in the valuation multipliers:

n	$\Delta\frac{1}{2}(A_{x\overline{n}\vert} + A_{x\overline{n}\vert}^{\;1})$	$\Delta \mathbf{a}_{x\overline{n}\vert}$
11	·0025	·029
16	·0023	·041
21	·0019	·050
26	·0016	·052
31	·0013	·054

It will be observed that while $\Delta \mathbf{a}_{x\overline{n}\vert}$ increases with n $\Delta\frac{1}{2}(A_{x\overline{n}\vert} + A_{x\overline{n}\vert}^{\;1})$ decreases, but a counteracting factor is furnished by the relative numerical decrease in the Net Premiums at the longer future durations. For example, if the amount payable at maturity under policies in force at each of the future durations 11, 21 and 31 is 10,000 the respective Net Premiums in force are not likely to be more than 500, 300 and 200. On these figures an alteration of the mean maturing age from 56 to 55 would increase the valuation as follows:

n	Increase in Value of	
	Sum Assured	Net Premiums
11	25	15
21	19	15
31	13	11

On the whole it is fairly safe to say that a decrease in r would cause an increase in the value of Sum Assured at least half as large again as in the Premiums, so that if r is the value at which the errors in Sum Assured and Premiums balance then $\overline{q-r}$ would have to be at least half as great again as $\overline{p-r}$. If p is not less than 1·075 and q is not greater than 1·09, we deduce that r is not less than 1·045.

17. There are so many factors involved in this estimate that one would prefer to use the method of paragraph 12

where Sum Assured and Premiums are treated separately. Moreover, the latter method is more convenient in that the same tables of Z can be employed as are already in use for Endowment Assurances.

18. Taking r as low as unity for both Sum Assured and Premiums, we have a simple method with a safety margin and the error when this is done will now be investigated. Putting $r=1$ is equivalent to taking the arithmetic mean of the maturing ages as can be seen by remembering that

$$M = \frac{\log \Sigma S_x r^x - \log \Sigma S_x}{\log r}$$

the limit of which as r approaches unity is

$$\frac{\Sigma S_x x}{\Sigma S_x}$$

19. If y_0 is the mean maturing age on the basis $r=1$, y_1 the mean age obtained by weighting with c and y_2 by weighting with c^2, &c., then the adoption of age y_0 is equivalent to the substitution of the series

$$k_0 + k_1 c^{y_0} + k_2 c^{2y_0} +, \text{ \&c.}$$

for

$$k_0 + k_1 c^{y_1} + k_2 c^{2y_2} +, \text{ \&c.},$$

the error being

$$k_1 c^{y_0}(1 - c^{y_1 - y_0}) + k_2 c^{2y_0}(1 - c^{2y_2 - 2y_0}) +, \text{ \&c.}$$

It was found by trial that $y_1 - y_0$, $y_2 - y_1$, and $y_3 - y_2$, are in descending numerical order, feasible values in the case of a wide range being 3, 2·3 and 1·6 respectively. Assuming these values we have:

$$1 - c^{y_1 - y_0} = -\cdot 3151$$
$$1 - c^{2y_2 - 2y_0} = -1\cdot 6324$$
$$1 - c^{3y_3 - 2y_0} = -5\cdot 6206$$

and the total error will be the second term in the expansion multiplied by $-\cdot 3151$, plus the third by $-1\cdot 6324$, plus the fourth by $-5\cdot 6206$, further terms being negligible. Putting y_0 at 56, the figures for the Sum Assured would be:

n	Second Term	Third Term	Fourth Term	Total
11	+ ·0102	− ·0031	+ ·0005	+ ·0076
21	+ ·0083	− ·0037	+ ·0007	+ ·0053
31	+ ·0057	− ·0031	+ ·0006	+ ·0032

and for the Net Premiums

n	Second Term	Third Term	Fourth Term	Total
11	+ ·11	− ·02	+ ·00	+ ·09
21	+ ·19	− ·04	+ ·01	+ ·16
31	+ ·20	− ·04	+ ·01	+ ·17

For 10,000 in force at each duration with Net Premiums of 500, 300 and 200 respectively as before, the actual errors would be, say,

n	Error in Value of	
	Sum Assured	Net Premium
11	+ 76	+ 45
21	+ 53	+ 48
31	+ 32	+ 34

20. On the alternative assumption that

$$y_0 = 61$$
$$y_1 - y_0 = 2{\cdot}0$$
$$y_2 - y_1 = 1{\cdot}5$$
$$y_3 - y_2 = 1{\cdot}0$$

the multipliers for the second, third and fourth terms are: −·2004, −·8950 and −2·4306 respectively, the figures for Sum Assured being:

n	Second Term	Third Term	Fourth Term	Total
11	+ ·0102	− ·0043	+ ·0007	+ ·0066
21	+ ·0083	− .0051	+ ·0011	+ ·0043
31	+ ·0057	− ·0042	+ ·0011	+ ·0026

and for the Net Premiums:

n	Second Term	Third Term	Fourth Term	Total
11	+ ·11	− ·02	+ ·00	+ ·09
21	+ ·19	− ·05	+ ·01	+ ·15
31	+ ·20	− ·05	+ ·01	+ ·16

Assuming the same amounts of Sum Assured and Net Premium as before the actual errors would be, say:

n	Error in Value of	
	Sum Assured	Net Premium
11	+ 66	+ 45
21	+ 43	+ 45
31	+ 26	+ 32

Again it appears that the basis $r=1$ would leave a considerable safety margin.

21. The estimates in the two preceding paragraphs can be compared with those of paragraph 15 as follows:

Roughly,
$$\frac{y_r - y_0}{y_1 - y_0} = \frac{r-1}{c-1},$$

so if $y_1 - y_0 = 3$, we have $y_{1\cdot075} - y_0 = 2\cdot354$ and $y_{1\cdot09} - y_0 = 2\cdot823$. Hence, if y_0 is 56, the adoption of y_0 in place of $y_{1\cdot075}$ and $y_{1\cdot09}$ respectively, would cause errors as follows:

n	Error in	
	$\frac{1}{2}(A_{\overline{xn}} + A_{\overline{xn}\rceil}^{\;1})$	$a_{\overline{xn}\rceil}$
11	·0070	·10
21	·0053	·17
31	·0035	·17

which compares reasonably well with the figures of paragraph 19. Again, if $y_1 - y_0 = 2$, we get $y_{1\cdot075} - y_0 = 1\cdot569$, and $y_{1\cdot09} - y_0 = 1\cdot882$, and, taking y_0 at 61, the errors are:

n	Error in	
	$\frac{1}{2}(A_{\overline{xn}} + A_{\overline{xn}\rceil}^{\;1})$	$a_{\overline{xn}\rceil}$
11	·0065	·09
21	·0048	·16
31	·0031	·17

which are reasonably close to the figures of paragraph 20.

22. Valuations of a Double Endowment class were made by the H^M Table at 3 per-cent interest, and the following table shows the results (1) on the basis $r=c$, (2) on the same basis with the suggested adjustment of ·001 of the endowment benefit, (3) on the basis $r=1·075$.

	Particulars	Exact Valuation	Excess over Exact Valuation of Valuation on basis		
			(1)	(2)	(3)
Sum Endowed	2,981,757	1,499,306	−1,836	+1,146	+434
Reversionary Bonus	180,968	104,261	− 184	− 3	− 13
Net Premium	79,969	1,049,354	−1,396	−1,396	+ 23
Reserve		554,213	− 624	+2,539	+398

This was not at all an unusually favourable case as the range of maturing ages was wide, the difference between mean ages based on c and c^2 being as high as 2·2 in some groups. M was about 55 over the body of the table, increasing gradually after future duration 30 to about 60.

23. The error in the value of Sum Assured is about what would be expected from the foregoing analysis, but the error in the value of Premiums when $r=c$ slightly exceeds Mr. Lidstone's estimated maximum of ·015. The reason for this was found in the fact that a considerable proportion of the business was at young ages. For the purpose of verifying this conclusion and also of obtaining a guide to the incidence of the error under a table that follows Makeham's law throughout, such as the $O^{M(5)}$, a valuation was made of the same business but excluding all cases under age 25 with the following result:

	Particulars	Exact Valuation	Excess over Exact Valuation of Valuation on basis		
			(1)	(2)	(3)
Sum Endowed	2,214,792	1,180,839	−1,191	+1,024	+409
Reversionary Bonus	162,169	95,941	− 305	− 143	−169
Net Premium	63,255	768,004	+ 29	+ 29	+904
Reserve		508,776	−1,525	+ 852	−664

The smallness of the error in the value of Net Premium here when $r=c$ appears to be due to the difference in distribution of the Sum Assured and Premiums.

24. It will be seen that the results confirm the conclusion that a Lidstone valuation taking $r=c$ with an adjustment equal to ·001 of the amount payable at maturity will give an approximation sufficient for any practical purpose with a safety margin. No valuation has been made with $r=$unity, but inspection of the above results makes it evident that this method would have given quite satisfactory results in this case, though with a wider margin.

25. In practical valuation work the question arises as to how many decimal places are to be retained in the various functions. It will doubtless be agreed that, although in a theoretical investigation to determine the trend and extent of error in any method exactness is necessary, there is no object in carrying such niceties into practical valuation work. Three significant figures are no doubt enough to retain in the valuation multipliers and if this is done it seems superfluous to take the mean age to a decimal. The valuations quoted in paragraph 21 were made with M correct to one decimal place, the multiplier for Sum Asured to five, and the annuity to three. Valuations were made by the same methods, taking M to the nearest integer, the Sum Assured multiplier to three places, and the annuity to one, and it may be of interest to quote the differences caused by the curtailment. They are as follows:

Increase in exact valuation	+560
„ valuation on basis $r=c$	−420
„ valuation on basis $r=1{\cdot}075$	+353

If the mean age is taken to the nearest integer and calculated by a table of logarithms to the base c as suggested by Mr. E. C. Coote, A.I.A. (*J.I.A.*, vol. xlii. p. 213), Mr. Lidstone's method is a very convenient one indeed for either Endowment or Double Endowment Assurances, requiring the insertion of only one extra function on the valuation card and the provision of one additional column in the valuation schedule.

TABLE A.

Showing the values of the first four terms when $\frac{1}{2}(A_{x\overline{n|}} + A_{x\overline{n|}}^{\;1})$ is expanded in powers of c^x for the Text-Book Table at 3 per-cent interest.

x	n	k_0 (1)	k_1c^x (2)	k_2c^{2x} (3)	k_3c^{3x} (4)	(1) + (2) + (3) + (4) (5)	$\frac{1}{2}(A_{x\overline{n|}} + A_{x\overline{n|}}^{\;1})$ (6)
15	11	·70260	−·00208	+·00001	−·00000	·70053	·70409
20	...	...	−·00329	+·00002	−·00000	·69933	·70002
25	...	...	−·00519	+·00006	−·00000	·69747	·69748
30	...	...	−·00821	+·00012	−·00000	·69451	·69453
35	...	...	−·01295	+·00030	−·00000	·68995	·68995
40	...	...	−·02044	+·00077	−·00002	·68291	·68292
45	...	...	−·03227	+·00191	−·00008	·67216	·67223
50	...	...	−·05094	+·00477	−·00028	·65615	·65613
55	...	...	−·08043	+·01190	−·00112	·63295	·63308
60	...	...	−·12697	+·02966	−·00442	·60087	·60124
15	21	·51706	−·00424	+·00006	−·00000	·51288	·51377
20	...	...	−·00670	+·00015	−·00000	·51051	·51064
25	...	...	−·01057	+·00037	−·00001	·50685	·50686
30	...	...	−·01670	+·00092	−·00004	·50124	·50126
35	...	...	−·02635	+·00229	−·00012	·49288	·49291
40	...	...	−·04160	+·00572	−·00047	·48071	·48072
45	...	...	−·06566	+·01424	−·00186	·46378	·46398
50	...	...	−·10366	+·03547	−·00731	·44156	·44258
15	31	·38728	−·00720	+·00030	−·00001	·38037	·37938
20	...	...	−·01136	+·00075	−·00003	·37664	·37639
25	...	...	−·01794	+·00190	−·00011	·37113	·37112
30	...	...	−·02832	+·00474	−·00044	·36326	·36323
35	...	...	−·04472	+·01180	−·00172	·35264	·35274
40	...	...	−·07058	+·02941	−·00675	·33936	·34013
15	41	·29654	−·01179	+·00140	−·00009	·28606	·28367
20	...	...	−·01862	+·00348	−·00034	·28106	·28051
25	...	...	−·02939	+·00866	−·00134	·27447	·27456
30	...	...	−·04640	+·02158	−·00524	·26648	·26726

TABLE B.

Showing the values of the first four terms when $a_{x\overline{n}|}$ is expanded in powers of c^x for the Text-Book Table at 3 per-cent interest.

x	n	l_0 (1)	l_1c^x (2)	l_2c^{2x} (3)	l_3c^{3x} (4)	(1) + (2) + (3) + (4) (5)	$a_{x\overline{n}|}$ (6)
15	10	8·258	− ·023	+ ·000	− ·000	8·235	8·320
20	...	...	− ·036	+ ·000	− ·000	8·222	8·243
25	...	...	− ·057	+ ·000	− ·000	8·201	8·203
30	...	...	− ·089	+ ·001	− ·000	8·170	8·170
35	...	...	− ·141	+ ·002	− ·000	8·119	8·119
40	...	...	− ·223	+ ·004	− ·000	8·039	8·040
45	...	...	− ·352	+ ·011	− ·000	7·917	7·917
50	...	...	− ·555	+ ·027	− ·001	7·729	7·729
55	...	...	− ·876	+ ·067	− ·004	7·445	7·445
60	...	...	−1·383	+ ·166	− ·016	7·025	7·027
15	20	14·034	− ·098	+ ·001	− ·000	13·937	14·103
20	...	...	− ·155	+ ·001	− ·000	13·880	13·919
25	...	...	− ·244	+ ·004	− ·000	13·794	13·796
30	...	...	− ·385	+ ·009	− ·000	13·658	13·658
35	...	...	− ·608	+ ·023	− ·001	13·448	13·448
40	...	...	− ·960	+ ·057	− ·003	13·128	13·128
45	...	...	−1·516	+ ·142	− ·011	12·649	12·650
50	...	...	−2·393	+ ·355	− ·044	11·952	11·956
15	30	18·074	− ·254	+ ·004	− ·000	17·824	18·046
20	...	...	− ·401	+ ·010	− ·000	17·683	17·731
25	...	...	− ·633	+ ·023	− ·001	17·463	17·467
30	...	...	−1·000	+ ·057	− ·003	17·128	17·132
35	...	...	−1·578	+ ·143	− ·013	16·626	16·633
40	...	...	−2·491	+ ·356	− ·049	15·890	15·911
15	40	20·899	− ·544	+ ·020	− ·001	20·374	20·634
20	...	...	− ·858	+ ·049	− ·003	20·087	20·144
25	...	...	−1·355	+ ·123	− ·010	19·657	19·661
30	...	...	−2·139	+ ·307	− ·041	19·026	19·031

TABLE C.

Showing the values of the first four terms when A_{xn} is expanded in powers of e^x for the Text-Book Table at 3 per-cent interest.

x	n	m_0 (1)	$m_1 e^x$ (2)	$m_2 e^{2x}$ (3)	$m_3 e^{3x}$ (4)	(1) + (2) + (3) + (4) (5)	$A_{\overline{xn}\vert}$ (6)
5	11	·73035	+ ·00066	– ·00000	+ ·00000	·73101	·72854
0	...	...	+ ·00104	– ·00000	+ ·00000	·73139	·73078
5	...	...	+ ·00166	– ·00000	+ ·00000	·73201	·73195
0	...	...	+ ·00259	– ·00003	+ ·00000	·73291	·73291
5	...	...	+ ·00411	– ·00006	+ ·00000	·73440	·73440
0	...	...	+ ·00650	– ·00012	+ ·00000	·73673	·73670
5	...	...	+ ·01025	– ·00032	+ ·00000	·74028	·74029
0	...	...	+ ·01617	– ·00079	+ ·00003	·74576	·74576
5	...	...	+ ·02552	– ·00195	+ ·00012	·75404	·75402
0	...	...	+ ·04028	– ·00484	+ ·00047	·76626	·76621
5	21	·56211	+ ·00285	– ·00002	+ ·00000	·56494	·56010
0	...	...	+ ·00450	– ·00004	+ ·00000	·56657	·56547
5	...	...	+ ·00711	– ·00011	+ ·00000	·56911	·56905
0	...	...	+ ·01121	– ·00026	+ ·00000	·57306	·57307
5	...	...	+ ·01771	– ·00067	+ ·00003	·57918	·57919
0	...	...	+ ·02796	– ·00166	+ ·00009	·58850	·58851
5	...	...	+ ·04416	– ·00414	+ ·00032	·60245	·60243
0	...	...	+ ·06970	– ·01034	+ ·00128	·62275	·62265
5	31	·44444	+ ·00740	– ·00011	+ ·00000	·45173	·44527
0	...	...	+ ·01168	– ·00028	+ ·00001	·45585	·45444
5	...	...	+ ·01844	– ·00067	+ ·00003	·46224	·46213
0	...	...	+ ·02913	– ·00166	+ ·00009	·47200	·47188
5	...	...	+ ·04596	– ·00417	+ ·00038	·48661	·48641
0	...	...	+ ·07256	– ·01037	+ ·00143	·50806	·50745
5	41	·36217	+ ·01584	– ·00058	+ ·00002	·37745	·36988
0	...	...	+ ·02500	– ·00144	+ ·00008	·38581	·38415
5	...	...	+ ·03947	– ·00358	+ ·00029	·39835	·39822
0	...	...	+ ·06230	– ·00894	+ ·00119	·41672	·41657

Notes on Finite Differences. *By* DUNCAN C. FRASER, M.A., F.I.A.

1. THE publication of these Notes has been suggested by the perusal of the new edition of Messrs. Burn and Brown's well-known manual.* It is not, however, our intention to offer anything in the nature of criticism of this work. The Authors have had an exceptional experience of the needs of actuarial students, and our congratulations are due to them on the appearance of the second edition of a book from which so many members of the Institute have gained their first ideas of the calculus of Finite Differences and its applications.

2. The amount of literature that is readily accessible on the subject of Finite Differences is not very extensive. It is a misfortune that the excellent Treatise of Professor Boole, the successive editions of which appeared in 1860, 1872, and 1880, has not been revised and brought up to date. But we have the satisfaction of knowing that the eminent authority on this subject, Dr. W. F. Sheppard, has been giving special attention to the actuarial applications of the calculus, and we have welcomed the appearance of two most valuable and illuminating papers by him in the *Journal* (*J.I.A.*, xlviii, 171 and 390). Most of Dr. Sheppard's work is scattered through the volumes of the London Mathematical Society, the Statistical Society, and Biometrika. But the attention of students may be especially drawn to his articles in the Encyclopedia Britannica on Differences and on Interpolation which contain in brief space a very convenient synopsis of these subjects.

3. The calculus of Finite Differences is an attractive and simple branch of mathematics, and among the propositions which are of most interest and use to actuaries there is probably not one which is beyond the comprehension of a student who has mastered the binomial theorem. But the very simplicity of the subject has led to its being treated in a brief and summary manner in mathematical text books, which generally introduce it at a stage when the knowledge assumed to have been acquired is of a kind and degree outside the range of the average actuarial student, and not really necessary to the exposition. A student who wishes to go beyond the mere elements often

* Elements of Finite Differences, also Solutions to Questions set for Part I of the Examinations of the Institute of Actuaries. By J. BURN, F.I.A., and E. H. BROWN, F.I.A. Second Edition. P. 289. Price 10*s*. 6*d*. net. London: C. & E. Layton. 1915.

finds himself involved in difficulties which are not inherent in the subject, but arise from the manner in which it is presented, and it may not be without advantage to take up a few points and discuss them, on a quite elementary basis, rather more fully and freely than is usually done.

4. The special usefulness of the formulas of Central Differences for purposes of interpolation is not always fully appreciated by students, and perhaps the best way of suggesting their advantages is to work out a normal example in detail. Taking a case which can be easily tested, the value of $(1+i)^{22}$ when $i = \cdot 05$ has been calculated by an ordinary descending difference formula, by Stirling's formula, and by Bessel's formula. For the first and second formulas seven values of the function have been employed

$$u_{-3}=(1+i)^{5};\quad u_{-2}=(1+i)^{10};\quad \ldots\quad u_{3}=(1+i)^{35}$$

and the calculation is carried in each case to $\Delta^6 u_{-3}$; for the 3rd formula the value of u_4 has also been employed, so that the mean 6th difference might be calculated.

5. The formulas and details of the arithmetical work are given in Note (I). Below is given the error at each stage of the work, the error being taken as (approximate value) − (true value).

Order of Difference	Descending Differences	Stirling's Formula	Bessel's Formula
	(Approximation) − (True Value)		
1	− ·450,094,739	− ·010,477,663	+ ·021,259,879
2	− ·052,619,146	+ ·002,217,354	− ·0[illegible]0,413,195
3	− ·001,372,063	+ ·000,027,916	− ·000,237,825
4	+ ·000,043,799	− ·000,025,232	+ ·000,004,136
5	− ·000,003,142	− ·000,000,096	+ ·000,002,961
6	+ ·000,000,316	+ ·000,00[illegible],316	− ·000,000,049

6. Comparing the results of Stirling's formula with those of the descending difference formula, it is seen that while the final values are identical, as they should be, since the formulas proceed to the same final difference, the central difference formula has the great advantage that it keeps closer to the true value at each stage of the work before the final difference is reached. A similar comparison can be made of the descending difference formula with Bessel's formula, but here there is no coincidence unless the work is carried to the seventh difference.

7. In the above comparison the work has been carried to sixth differences and the descending difference formula has been selected whose final result is identical with that of one central difference formula and in close agreement with that of the other. But if it should be intended to stop at an order of differences, short of the sixth, another descending difference formula would ordinarily be taken, and therefore the descending difference formulas whose initial terms are u_{-2}, u_{-1}, and u_0 have been examined ; and further the ascending difference formulas whose initial terms are u_1, u_2, and u_3, have also been tested.

Without giving arithmetical details the results of comparisons with the two central difference formulas may be stated in tabular form.

The descending difference formula whose initial term is

u_{-3}	u_{-2}	u_{-1}	u_0

and the ascending difference formula whose initial term is

u_3	u_2	u_1	u_0

agree identically with Stirling's formula whose initial term is u_0, if carried to the term involving

$\Delta^6 u_{-3}$	$\Delta^4 u_{-2}$	$\Delta^2 u_{-1}$	u_0

The descending difference formula whose initial term is that given above, and the ascending difference formula whose initial term is

u_4	u_3	u_2	u_1

agree identically with Bessel's formula whose initial term is $\frac{1}{2}(u_0+u_1)$, if carried to the term involving

$\Delta^7 u_{-3}$	$\Delta^5 u_{-2}$	$\Delta^3 u_{-1}$	Δu_0

But in all cases, if the calculation stops at any other term. before or after the term where there is agreement, then the central difference formula gives a better result, whether Stirling's or Bessel's formula be taken.

8. We may say, therefore, that whichever central difference formula is taken, we gain the advantage of combining in a compact form in a single formula the best results of several descending and ascending difference formulas.

9. The numerical example given in detail above has been so chosen that, except in the last term of Stirling's

formula, the central differences are larger than the descending differences. It will be understood, therefore, that the advantage of a central difference formula is gained by the use of smaller coefficients. An examination of the figures given in Note I brings this out very clearly. And it can be stated generally that any formula which goes outside the track of the central differences introduces coefficients which are larger than the corresponding coefficients of a central difference formula, and the farther the formula departs from central differences the more does the relative magnitude of the coefficients increase.

10. It remains to compare the two central difference formulas with one another, and a close examination of the errors leads to an interesting point. In both formulas the best results are obtained by stopping at mean differences; that is at odd differences in Stirling's formula, and at even differences in Bessel's formula. Thus in Stirling's formula the result to third differences is practically as good as that to fourth differences, and in the particular example taken, it is better to stop at fifth differences than at sixth differences. An elementary investigation of this feature is given in Note II. It is also pointed out in the same note that Gauss's formula agrees with Stirling's formula to even differences, and with Bessel's formula to odd differences, and can be adapted by a simple device to agree with either of these formulas to mean differences. Thus Gauss's formula appears in normal cases to combine the advantages of all the other formulas, whether they proceed by ascending or by descending or by central differences.

11. The statement made above that in using Stirling's formula the best relative results are usually obtained by stopping at a mean odd difference may appear to be in conflict with a remark made by Dr. W. F. Sheppard that in the case of Stirling's formula it is important to end at an even difference. But the contradiction is one in appearance only. In Dr. Sheppard's paper, "On the Accuracy of Interpolation by Finite Differences," which appeared in the proceedings of the London Mathematical Society (Series 2, vol. 4, parts 4 and 5), he deals with cases where the tabular error, that is the error due to the fact that the tabulated values are themselves only approximate, is of greater practical importance than the error due to omitting terms of the formula. His investigations show that if the possible error in the tabulated values is $\pm\cdot5$ in the last figure, the possible error in the calculation of an interpolated value u_n is diminished in numerical value

by proceeding to the term involving $\Delta^{2r}u_{-r}$, instead of stopping at the preceding mean odd difference, by the quantity

$$\frac{n^2(1^2-n^2)(2^2-n^2)\ldots(\overline{r-1}^2-n^2)}{(\underline{r})^2}\times\cdot 5$$

We may assume that n does not exceed $\cdot 5$. This expression is then less than $\frac{n^2}{r^2}\times\cdot 5$ and its value is less than $\cdot 125$ if we stop at second differences, and less than $\cdot 03125$ if we stop at fourth differences. The case we have been discussing, however, is one where it is not necessary to obtain a result to as many significant figures as are contained in the tabular values, and where in consequence the error due to omitting terms of the formula is of greater importance than the tabular error; and in such a case our investigation shows how the degree of accuracy to which we are working may be readily judged.

12. Professor Everett has given (*J.I.A.*, xxxv, 452) a central difference formula which may be written

$$u_p = pu_1 + \frac{p(p^2-1)}{\underline{3}}\Delta^2 u_0 + \frac{p(p^2-1)(p^2-4)}{\underline{5}}\Delta^4 u_{-1} + \text{, \&c.}$$

$$+ qu_0 + \frac{q(q^2-1)}{\underline{3}}\Delta^2 u_{-1} + \frac{q(q^2-1)(q^2-4)}{\underline{5}}\Delta^4 u_{-2} + \text{, \&c.,}$$

where $p+q=1$.

If the construction of the formula be carefully examined it will be found that when taken to any even order of differences it is identical with Bessel's formula taken to the next order of odd differences (see *J. I.A.*, xliii, 240).

Everett's formula can be written

$$u_p = F(p)u_1 - F(p-1)u_0$$

When p is an integer a meaning can be given to each of the functions on the right. $F(p)u_1$ is the sum of p alternate u's, of which u_1 is the central term; and $F(p-1)u_0$ is the sum of $p-1$ alternate u's, of which u_0 is the central term.

Thus $$F(p)u_1 = u_p + u_{p-2} + u_{p-4} + \ldots + u_{-p+2}$$

$$F(p-1)u_0 = u_{p-2} + u_{p-4} + \ldots + u_{-p+2}$$

Here $(p-1)$ of the values are common to the two functions, and disappear when the difference is taken, leaving as the result the value u_p.

13. We have been discussing formulas which are associated with the names of Stirling, Bessel, Gauss, and Everett; but all these formulas are in fact due to Newton, or are transformations of Newton's formulas. Newton's descending difference formula was given in the 5th Lemma of the 3rd Book of the Principia (1687). Central Difference formulas derived from an odd, and from an even number of values of the function, will be found in the "Methodus Differentialis," a small tract by Newton on Interpolation, printed for the first time in 1711 in a volume entitled "Analysis per Quantitatum Series, Fluxiones ac Differentias," &c. This volume which is a collection of various writings of Newton, edited by William Jones, is in the Institute library. The whole of Newton's work on Interpolation would occupy less than a dozen pages of the *Journal*, and it might be worth consideration whether it should not be translated and inserted. It should be remarked that in each case Newton, after giving formulas which apply when the values of the function are taken at equal intervals, adds formulas for unequal intervals.

14. Stirling adopted the title of Newton's tract, "Methodus Differentialis," for the work in which is given his account of the two central difference formulas, one being equivalent to Bessel's formula. The first edition appeared in 1730. A copy of the 2nd edition (1764) will be found in the library, and the following is a more or less approximate translation of the remarks which he appends at page 107 to his statements of the formulas:

"Many celebrated mathematicians after Newton have "discussed the question of describing a curve of the parabolic "order through given points. But all their solutions are the "same as those given above, which hardly differ from those of "Newton in the 5th Lemma of the 3rd Book of the Principia, "and in the Methodus Differentialis, edited by Mr. Jones. "Newton describes a parabolic curve through given points; "others have considered the determination of the terms from "different data; but, however it may be expressed, and however "it may be carried out, it is the same problem. And certainly "the discovery of the mathematical forms for the values of the "interpolated ordinate is exceedingly ingenious and worthy "of the celebrity of its author; but after the mathematical "forms are given the investigation of the problem is easy, "nothing more being required than the solution of simple "equations."

15. The good sense and the modesty of this note will be

admitted. But it is only fair to say that, although Newton's tract was published in 1711, and was probably written much earlier, it is to Stirling's volume, referred to above, that the central difference formula to which his name has been attached owes the general recognition of its value and usefulness. Stirling's Treatise was a work of considerable reputation in the 18th century, and is the fount and origin of questions of the type with which we have become so familiar in examination papers: What is the sum to n terms of the series $1.2.4+2.3.5+3.4.6+\ldots$? These questions are of little importance out of the examination room, at any rate to Actuaries, except in the very simplest cases. A special exception may be made of the binomial coefficients, which deserve even closer study than has yet been given them, for it may be maintained that it is simply in the properties of the binomial coefficients that the explanation of the value of central difference formulas is to be found. May we venture to hope that the tortuous and sterile ingenuities which have racked the brains of examiners and of examinees for so long will be allowed to sink into a quiet oblivion?

16. Newton's method of interpolation is based on the principle of representing the given values of the function by ordinates, and finding a curve of the parabolic order which passes through the extremities of the ordinates. A curve of the parabolic order is a curve whose equation can be expressed in the form

$$y=a+bx+cx^2+dx^3+, \text{\&c.}$$

the number of terms on the right being finite; and the particular curve used in each case is the curve of lowest degree in x which passes through the extremities of the ordinates employed. All the formulas based on the Newtonian principle represent the same curve if they employ the same values of the function, and will therefore give the same interpolated value. The final difference which appears in any formula determines exactly and completely what values of the function have been used. Thus, if the final difference is $\Delta^4 u_{-2}$, the values of the function employed are u_{-2}, u_{-1}, u_0, u_1, u_2, and no other values are used. This is so whether the initial term is u_{-2}, or u_{-1}, or u_0, or u_1, or u_2, and whatever differences are used between the initial term and the final difference. All formulas which have the same final difference are in fact identities. This principle, which has been discussed from a different point of view in *J.I.A.*, vol. xliii, p. 238, is important in the practical use of interpolation formulas. Having fixed in

any particular case on the final difference, there is a considerable range of choice as to the formula which may be employed; and if differences beyond that selected as the final difference are known there is the further advantage that the error can be approximately calculated beforehand, and we then know to what degree of accuracy we are working. In any formula the error, whether its magnitude can be estimated or not, depends on the last difference employed.

17. Dr. W. F. Sheppard has introduced with special reference to central difference formulas a scheme of notation based on the operators δ and μ where $\delta u_0 = u_{\frac{1}{2}} - u_{-\frac{1}{2}}$ and $\mu u_0 = \frac{1}{2}(u_{\frac{1}{2}} + u_{-\frac{1}{2}})$. The notation is elegant and attractive, and has been fully explained by Dr. Buchanan (*J.I.A.*, vol. xlii, p. 370); but some experience of actuarial students leads to the conviction that the multiplication of symbols is hampering and confusing to them. It may be suggested therefore that in elementary work it would be unwise to introduce the new notation, and that it should be reserved for post-graduate studies.

18. It is curious that no convenient analytical proof of the central difference formulas has yet been placed before our students. Dr. W. F. Sheppard has given in the Paper already referred to, alternative proofs of Gauss's formula, which are of a fairly simple character. Mr. Robert Henderson has given interesting proofs of Stirling's and Bessel's formulas based on the use of *sinh* and *cosh* functions (Trans. Act. Soc. of America, ix, 211); but seeing that we exclude—and probably rightly exclude—the *sin* and *cos* functions from the syllabus of our examinations, it seems to follow that we must consider the *sinh* and *cosh* functions to be also excluded. There is accordingly a gap which waits to be filled. At the same time excessive importance should not be attached, from the point of view of the education of our students, to the production of analytical proofs. In the history of mathematics it has often happened that (1) an important proposition has been announced with certain arguments which suggest its truth; (2) its value and truth have been proved in practical use; (3) analytical proofs have then been given; (4) the proofs have been proved to be imperfect, and further proofs are given with elaborate investigations of the conditions under which they are valid. On the principle that the development of the individual recapitulates the development of the race, it is often the wisest course to follow

the historical stages in dealing with actuarial students, and of the steps enumerated above the first two are the most vitally important, and it is frequently quite unnecessary to go beyond them. The analytical proofs can be left for students who have a passion for such things, and it is to these students that we look for the extension and systematization of our knowledge.

NOTE I.—*Numerical example of Interpolation.*

The value of $(1+i)^{22}$ where $i = \cdot 05$ is to be calculated by interpolation between the values of

$$(1+i)^5,\ (1+i)^{10},\ \ldots (1+i)^{35},\ (1+i)^{40}$$

The formulas used are given below. Seven values of the function are employed in the descending difference formula and in Stirling's formula, while the eighth value is employed in Bessel's formula to obtain the mean sixth difference. The coefficients of the formulas are all written positively, so that each coefficient is preceded by its correct sign. For Stirling's and Bessel's formulas compare *J.I.A.*, vol. xliii. p. 238-240.

Ordinary Descending Difference Formula:

$$(1+i)^{22} = u_n = u_{-3+3+n}$$

$$= u_{-3} + (3+n)\Delta u_{-3} + \frac{(2+n)(3+n)}{2}\Delta^2 u_{-3}$$

$$+ \frac{(1+n)(2+n)(3+n)}{6}\Delta^3 u_{-3} + \frac{n(1+n)(2+n)(3+n)}{24}\Delta^4 u_{-3}$$

$$- \frac{n(1^2-n^2)(2+n)(3+n)}{120}\Delta^5 u_{-3} + \frac{n(1^2-n^2)(2^2-n^2)(3+n)}{720}\Delta^6 u_{-3}\,. \quad (1$$

Stirling's Formula:

$$(1+i)^{22} = u_{0+n}$$

$$= u_0 + n\Delta\frac{u_0+u_{-1}}{2} + \frac{n^2}{2}\Delta^2 u_{-1}$$

$$- \frac{n(1^2-n^2)}{6}\Delta^3\frac{u_{-1}+u_{-2}}{2} - \frac{n^2(1^2-n^2)}{24}\Delta^4 u_{-2}$$

$$+ \frac{n(1^2-n^2)(2^2-n^2)}{120}\Delta^5\frac{u_{-2}+u_{-3}}{2} + \frac{n^2(1^2-n^2)(2^2-n^2)}{720}\Delta^6 u_{-3}\,. \quad . \quad . (2$$

Bessel's Formula:

$$(1+i)^{22} = u_{\cdot 5 - \cdot 5 - n}$$

$$= \frac{u_0 + u_1}{2} - (\cdot 5 - n)\Delta u_0 - \frac{n(1-n)}{2}\Delta^2 \frac{u_{-1}+u_0}{2}$$

$$+ \frac{n(\cdot 5 - n)(1-n)}{6}\Delta^3 u_{-1} + \frac{n(1^2-n^2)(2-n)}{24}\Delta^4 \frac{u_{-1}+u_{-2}}{2}$$

$$- \frac{n(\cdot 5 - n)(1^2-n^2)(2-n)}{120}\Delta^5 u_{-2} - \frac{n(1^2-n^2)(2^2-n^2)(3-n)}{720}$$

$$\Delta^6 \frac{u_{-2}+u_{-3}}{2} + \ . \quad . \quad . \quad (3$$

The following table gives the arithmetical details at each stage of the work.

Calculation of $(1+i)^{22}$ *at 5 per-cent from the values* $(1+i)^5$, $(1+i)^{10}$ . . . $(1+i)^{35}$.

True Value of $(1+i)^{22} = 2\cdot 925,260,720$, . . .

	Difference	Coefficient	Product	Approximation
	Ordinary descending difference formula. *Initial term* $u_{-3} = 1\cdot 276\ 281,562,5$			
1	·352,613,064,3	3·4	1·198,884,418,9	2·475,165,981
2	·097,420,488,3	4·08	·397,475,592,2	2·872,641,574
3	·026,915,484,8	1·904	·051,247,083,0	2·923,888,657
4	·007,436,252,2	·190,4	·001,415,862,4	2·925,304,519
5	·002,054,499,2	·022,848	– ·000,046,941,2	2·924,257,578
6	·000,567,620,6	·006,092,8	·000,003,458,4	2·925,261,036
	Stirling's formula. *Initial term* $u_0 = 2\cdot 653,297,705,1$			
1	·653,713,380,7	·4	·261,485,352,3	2·914,783,057
2	·158,687,710,1	·08	·012,695,016,8	2·927,478,074
3	·039,097,112,7	– ·056	– ·002,189,438,3	2·925,288,636
4	·009,490,751,4	– ·005,6	– ·000,053,148,2	2·925,235,488
5	·002,338,309,5	·010,752	·000,025,141,5	2·925,260,629
6	·000,567,620,6	·000,716,8	·000,000,406,8	2·925,261,036
	Bessel's formula. *Initial term* $(u_0 + u_1)/2 = 3\cdot 019,826,323,0$			
1	·733,057,235,8	– ·1	– ·073,305,723,6	2·946,520,599
2	·180,608,954,3	– ·12	– ·021,673,074,5	2·924,847,525
3	·043,842,488,4	·004	·000,175,370,0	2·925,022,895
4	·010,801,811,3	·022,4	·000,241,960,6	2·925,264,856
5	·002,622,119,8	– ·000,448	– ·000,001,174,7	2·925,263,681
6	·000,646,032,0	– ·004,659,2	– ·000,003,010,0	2·925,260,671

NOTE II.—*Investigation of the approximate error due to the omission of terms in a central difference formula.*

For this purpose it is convenient to use Gauss's formula which to sixth differences is

$$\begin{aligned} u_n = u_0 + n\Delta u_0 + n_2\Delta^2 u_{-1} \\ + (n+1)_3\Delta^3 u_{-1} + (n+1)_4\Delta^4 u_{-2} \\ + (n+2)_5\Delta^5 u_{-2} + (n+2)_6\Delta^6 u_{-3} \quad . \quad . \quad . \quad (4) \end{aligned}$$

In this formula all the differences are ordinary differences; and all the coefficients are ordinary binomial coefficients, the coefficients of $\Delta^{2r+1}u_{-r}$ and of $\Delta^{2r+2}u_{-r-1}$ being the same as those of x^{2r+1} and of x^{2r+2} in the expansion of $(1+x)^{n+r}$.

Gauss's formula can easily be shown to be identical with Stirling's formula to even differences, and with Bessel's formula to odd differences.

The coefficients which occur in the formula can be obtained by the continuous multiplication of the factors in the accompanying table, which also gives specimen values of the factors to two decimal places.

Factors of the coefficients in Gauss's formula.

Factor	Sign	Values				
n	+	·1	·2	·3	·4	·5
$(n-1)/2$	−	·45	·4	·35	·3	·25
$(n+1)/3$	+	·37	·4	·43	·45	·5
$(n-2)/4$	−	·47	·45	·42	·4	·37
$(n+2)/5$	+	·42	·44	·46	·48	·50
$(n-3)/6$	−	·48	·47	·45	·43	·42
$(n+3)/7$	+	·44	·46	·47	·49	·50
$(n-4)/8$	−	·49	·47	·46	·45	·44

We may suppose that n is not greater than ·5. In that case the factors, after the first two lines in the table, differ little from ·5, and never exceed that value. Also they are alternately $+ve$ and $-ve$.

It follows that, supposing the work to stop after fourth difference, the error due to omitting terms after that involving $\Delta^4 u_{-2}$ in Gauss's formula, or in Stirling's formula, is approximately the difference between $(n+1)_4 . \Delta^4 u_{-2}$ and the expression

$$(n+1)_4\Delta^4\left\{1+\frac{1}{2}\Delta-\frac{1}{4}\Delta^2E^{-1}-\frac{1}{8}\Delta^3E^{-2}+\frac{1}{16}\Delta^4E^{-2}+, \text{ \&c.}\right\}u_{-2} . (5)$$

so that the expression for the error involves the fifth and higher orders of differences.

Since $1+\frac{1}{2}\Delta=\frac{E+1}{2}$ the expression (5) can be transformed into

$$(u+1)_4 \,.\, \Delta^4\frac{E+1}{2}\left\{1-\frac{1}{4}\Delta^2E^{-1}+\frac{1}{16}\Delta^4E^{-2}+\ldots\right\}u_{-2} \quad . \quad . \quad . \quad (6)$$

and therefore, if in Gauss's formula we substitute the mean difference $\Delta^4\frac{u_{-2}+u_{-1}}{2}$ for Δ^4u_{-2} the approximate expression for the error involves the sixth and higher orders of difference; and it will be noticed that the coefficient of the sixth difference in (6) is less than that of the fifth difference in (5). The substitution, therefore, improves the accuracy of the formula in a sensible degree; but when the substitution is made it will be found that Gauss's formula then becomes identical with Bessel's formula to the fourth mean difference; and therefore Bessel's formula to the fourth mean difference is usually much more accurate than Stirling's formula to the fourth difference.

In the same way an expression for the error due to omitting the terms in Gauss's formula, or in Bessel's formula, after that involving Δ^5u_{-2} may be written down, and will be found to involve the sixth and higher order of differences. Using the relationship

$$1-\frac{1}{2}\Delta E^{-1}=\frac{1+E^{-1}}{2}$$

to transform the expression for the error, it can then be shown that the substitution of $\Delta^5\frac{u_{-2}+u_{-3}}{2}$ for Δ^5u_{-2} improves Gauss's formula in a sensible degree. But when this substitution is made Gauss's formula becomes identical with Stirling's formula to fifth differences, and therefore Stirling's formula to the fifth mean difference is usually much more accurate than Bessel's formula to the fifth difference.

The same argument applies if we stop at any other point in the formula provided we go to second differences at least. The results do not always to the same extent apply to the first mean difference, as will be readily understood by looking at the table of factors.

A comparison of the expression for the error in different cases will also indicate that in Stirling's formula the error involved in

stopping at a mean odd difference after the first is of the same order as that involved in stopping at the next even difference; and in Bessel's formula the error involved in stopping at a mean even difference is of the same order as that involved in stopping at the next odd difference.

All these results can be readily illustrated from the numerical example in the last note. Thus the error involved in stopping at the third mean difference in Stirling's formula should, according to our investigation, be represented approximately by

$$\cdot 25 \times (\text{coefficient of third mean difference}) \times (\text{fifth difference})$$

$$= \cdot 25 \times \cdot 056 \times \cdot 002 = \cdot 00003$$

and this agrees with the actual error.

We have only considered the case of n being $+ve$ and not greater than $\cdot 5$. If n is $-ve$, with a similar limitation of numerical value, Gauss's alternative formula can be employed, namely,

$$u_n = u_0 + n\Delta u_{-1} + (n+1)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-2} \ldots \quad . \quad . \quad . \quad (7)$$

The point to be remembered is that the best results are obtained by ending on a mean odd difference in line with u_0 in the table of differences, or by ending on a mean even difference which, if n is $+ve$, should be in line with the interval between u_0 and u_1, and if n is odd should be in line with the interval between u_0 and u_{-1}.

It is necessary to add that these rules may not be applicable if the differences of the order at which we stop show wide fluctuations. In such cases a small difference should be selected for the final term.

[With reference to par. 18 of the foregoing notes, the following analytical proofs of the various central difference formulas may be of interest. All the formulas are here obtained by a common process, namely, by (1) equating u_x or $(1+\Delta)^x u_0$, to a series consisting of the successive central differences with unknown coefficients; (2) expressing the central differences in terms of ordinary differences; and (3) determining the general coefficient by multiplying up by the appropriate powers of $1+\Delta$. We add demonstrations of two well-known formulas by the convenient method of operators.—Eds. *J.I.A.*]:

Central Difference Formulas.

1. *To express u_x in terms of ordinary Central Differences.*

Let
$$u_x = A_0 u_0 + A_1 a_0 + A_2 b_0 + A_3 c_0 + \ldots$$

Then, since

$$a_0 = \frac{1}{2}\Delta(u_{-1} + u_0) = \frac{\Delta}{1+\Delta}\left(1 + \frac{\Delta}{2}\right)u_0 \,; \quad b_0 = \frac{\Delta^2}{1+\Delta}u_0 \,;$$

$$c_0 = \frac{\Delta^3}{(1+\Delta)^2}\left(1 + \frac{\Delta}{2}\right)u_0 \,; \ \&c.$$

$$(1+\Delta)^x = A_0 + A_1\frac{\Delta}{1+\Delta} + \left(\frac{1}{2}A_1 + A_2\right)\frac{\Delta^2}{1+\Delta} + A_3\frac{\Delta^3}{(1+\Delta)^2} + \ldots$$

$$\ldots + \left(\frac{1}{2}A_{2r-1} + A_{2r}\right)\frac{\Delta^{2r}}{(1+\Delta)^r} + A_{2r+1}\frac{\Delta^{2r+1}}{(1+\Delta)^{r+1}} + \ldots$$

Multiplying up by $(1+\Delta)^r$, and equating coefficients of Δ^{2r},

$$\frac{1}{2}A_{2r-1} + A_{2r} = \frac{(r+x)(r+x-1)\ \ldots\ (x-r+1)}{2r\,!}$$

And, multiplying up by $(1+\Delta)^{r+1}$, and equating coefficients of Δ^{2r+1};

$$\frac{1}{2}A_{2r-1} + A_{2r} + A_{2r+1} = \frac{(r+x+1)(r+x)\ \ldots\ (x-r+1)}{(2r+1)\,!}$$

Hence, by subtraction,

$$A_{2r+1} = \frac{(r+x)(r+x-1)\ \ldots\ (x-r)}{(2r+1)\,!}$$

And since

$$\frac{1}{2}A_{2r-1} = \frac{1}{2}\cdot\frac{(r+x-1)(r+x-2)\ \ldots\ (x-r+1)}{(2r-1)\,!}$$

$$A_{2r} = \frac{x(r+x-1)(r+x-2)\ \ldots\ (x-r+1)}{2r\,!}$$

$$\therefore \qquad u_x = u_0 + xa_0 + \frac{x^2}{2\,!}b_0 + \frac{(x+1)x(x-1)}{3\,!}c_0$$
$$+ \frac{x(x+1)x(x-1)}{4\,!}d_0 + \ldots$$

2. *To express u_x in terms of Central Interval Differences.*

Let $$u_x = A_0 \frac{1}{2}(u_0+u_1) + A_1\alpha_0 + A_2\beta_0 + A_3\gamma_0 + \dots$$

Then, since

$$\frac{1}{2}(u_0+u_1) = \left(1+\frac{1}{2}\Delta\right)u_0\,; \quad \alpha_0 = \Delta u_0\,; \quad \beta_0 = \frac{\Delta^2}{1+\Delta}\left(1+\frac{1}{2}\Delta\right)u_0; \ \&c.$$

$$(1+\Delta)^x = A_0 + \left(\frac{1}{2}A_0 + A_1\right)\Delta + A_2\frac{\Delta^2}{1+\Delta} + \left(\frac{1}{2}A_2 + A_3\right)\frac{\Delta^3}{1+\Delta} + \dots$$
$$\dots + \left(\frac{1}{2}A_{2r-2} + A_{2r-1}\right)\frac{\Delta^{2r-1}}{(1+\Delta)^{r-1}} + A_{2r}\frac{\Delta^{2r}}{(1+\Delta)^r} + \dots$$

Multiplying up by $(1+\Delta)^{r-1}$, and equating coefficients of Δ^{2r-1},

$$\frac{1}{2}A_{2r-2} + A_{2r-1} = \frac{(r+x-1)(r+x-2)\dots(x-r+1)}{(2r-1)\,!}$$

And, multiplying up by $(1+\Delta)^r$, and equating coefficients of Δ^{2r},

$$\frac{1}{2}A_{2r-2} + A_{2r-1} + A_{2r} = \frac{(r+x)(r+x-1)\dots(x-r+1)}{2r\,!}$$

Hence, by subtraction,

$$A_{2r} = \frac{(r+x-1)(r+x-2)\dots(x-r)}{2r\,!}$$

And, since

$$\frac{1}{2}A_{2r-2} = \frac{1}{2}\cdot\frac{(r+x-2)(r+x-3)\dots(x-r+1)}{(2r-2)\,!}$$

$$A_{2r-1} = \frac{(2x-1)(r+x-2)(r+x-3)\dots(x-r+1)}{2(2r-1)\,!}$$

$$\therefore \qquad u_x = \frac{1}{2}(u_0+u_1) + \frac{2x-1}{2}\alpha_0 + \frac{x(x-1)}{2\,!}\beta_0$$
$$+ \frac{(2x-1)x((x-1)}{2\,.\,3\,!}\gamma_0$$
$$+ \frac{(x+1)x(x-1)(x-2)}{4\,!}\delta_0 + \dots$$

3. *To express u_x in terms of $\frac{1}{2}(u_{-\frac{1}{2}}+u_{\frac{1}{2}})$, $\Delta u_{-\frac{1}{2}}$, &c.*

Let

$$u_x = A_0\frac{1}{2}(u_{-\frac{1}{2}}+u_{\frac{1}{2}}) + A_1\Delta u_{-\frac{1}{2}} + A_2\Delta^2\frac{1}{2}(u_{-\frac{1}{2}}+u_{-\frac{1}{2}}) + \dots$$

Then, since

$$\frac{1}{2}(u_{-\frac{1}{2}}+u_{\frac{1}{2}}) = (1+\Delta)^{-\frac{1}{2}}\left(1+\frac{1}{2}\Delta\right)u_0:\ \ \Delta u_{-\frac{1}{2}} = (1+\Delta)^{-\frac{1}{2}}\Delta u_0:\ \&c.$$

$$(1+\Delta)^{x+\frac{1}{2}} = A_0 + \left(\frac{1}{2}A_0+A_1\right)\Delta + A_2\frac{\Delta^2}{1+\Delta} + \left(\frac{1}{2}A_2+A_3\right)\frac{\Delta^3}{1+\Delta} + \dots$$
$$+\left(\frac{1}{2}A_{2r-2}+A_{2r-1}\right)\frac{\Delta^{2r-1}}{(1+\Delta)^{r-1}} + A_{2r}\frac{\Delta^{2r}}{(1+\Delta)^r} + \dots$$

The only difference between this identity and the corresponding identity in (2) is that $(1+\Delta)^{x+\frac{1}{2}}$ replaces $(1+\Delta)^x$ on the left-hand side.

Hence, by following the same process as in (2) with the substitution of $x+\frac{1}{2}$ for x,

$$A_{2r} = \frac{\left(r+x-\frac{1}{2}\right)\left(r+x-\frac{3}{2}\right)\dots\left(x-r+\frac{1}{2}\right)}{2r\,!}$$

and

$$A_{2r-1} = \frac{x\left(r+x-\frac{3}{2}\right)\left(r+x-\frac{5}{2}\right)\dots\left(x-r+\frac{3}{2}\right)}{(2r-1)\,!}$$

$$\therefore\qquad u_x = \frac{1}{2}\left(u_{-\frac{1}{2}}+u_{+\frac{1}{2}}\right) + x\Delta u_{-\frac{1}{2}} + \frac{x^2-\frac{1}{4}}{2\,!}\,\frac{\Delta^2 u_{-\frac{1}{2}}+\Delta^2 u_{-\frac{1}{2}}}{2} + \dots$$

4. *To express u_x in terms of u_0, u_1 and their even Central Differences.*

Let $u_x = A_0u_0 + A_2b_0 + A_4d_0 + \dots$
$$+A_1u_1 + A_3b_1 + A_5d_1 + \dots$$

Then, since

$$b_0 = \frac{\Delta^2}{1+\Delta}u_0;\ d_0 = \frac{\Delta^4}{(1+\Delta)^2}u_0:\ u_1 = (1+\Delta)u_0;\ b_1 = \Delta^2u_0;\ \&c.,$$

$$(1+\Delta)^x = (A_0+A_1) + A_1\Delta + (A_2+A_3)\frac{\Delta^2}{1+\Delta} + A_3\frac{\Delta^3}{1+\Delta} + \dots$$
$$\dots + A_{2r-1}\frac{\Delta^{2r-1}}{(1+\Delta)^{r-1}} + (A_{2r}+A_{2r+1})\frac{\Delta^{2r}}{(1+\Delta)^r} + \dots$$

Multiplying up by $(1+\Delta)^{r-1}$, and equating coefficients of Δ^{2r-1},

$$A_{2r-1}=\frac{(r+x-1)(r+x-2)\ldots(x-r+1)}{(2r-1)!}$$

And, multiplying up by $(1+\Delta)^{r}$, and equating coefficients of Δ^{2r},

$$A_{2r-1}+A_{2r}+A_{2r+1}=\frac{(r+x)(r+x-1)\ldots(x-r+1)}{2r!}$$

Hence, by subtraction, since

$$A_{2r+1}=\frac{(r+x)(r+x-1)\ldots(x-r)}{(2r+1)!},$$

$$A_{2r}=\frac{(r+x-1)(r+x-2)\ldots(x-r)(1-x+r)}{(2r+1)!}$$

or, if ξ be written for $1-x$,

$$A_{2r}=\frac{(\xi+r)(\xi+r-1)\ldots(\xi-r)}{(2r+1)!}$$

$$\therefore \quad u_x=\xi u_0+\frac{(\xi+1)\xi(\xi-1)}{3!}b_0+\frac{(\xi+2)(\xi+1)\xi(\xi-1)(\xi-2)}{5!}d_0+\ldots$$

$$+xu_0+\frac{(x+1)x(x-1)}{3!}b_1+\frac{(x+2)(x+1)x(x-1)(x-2)}{5!}d_1+\ldots$$

5. To express u_x in terms of $u_0,\ \Delta u_0,\ b_0,\ \Delta^3 u_{-1},\ d_0,\ \ldots$

Let $\qquad u_x=A_0u_0+A_1\Delta u_0+A_2b_0+\ \ldots$

Then, since

$$b_0=\frac{\Delta^2}{1+\Delta}u_0;\quad \Delta^3u_{-1}=\frac{\Delta^3}{1+\Delta}u_0;\quad \&c.$$

$$(1+\Delta)^x=A_0+A_1\Delta+A_2\frac{\Delta^2}{1+\Delta}+A_3\frac{\Delta^3}{1+\Delta}+\ \ldots$$

$$+A_{2r-1}\frac{\Delta^{2r-1}}{(1+\Delta)^{r-1}}+A_{2r}\frac{\Delta^{2r}}{(1+\Delta^r)}+\ldots$$

Multiplying up by $(1+\Delta)^{r-1}$, and equating coefficients of Δ^{2r-1},

$$A_{2r-1}=\frac{(r+x-1)(r+x-2)\ldots(x-r+1)}{(2r-1)!}$$

And, multiplying up by $(1+\Delta)^r$, and equating coefficients of A^{2r},

$$A_{2r-1}+A_{2r}=\frac{(r+x)(r+x-1)\ldots(x-r+1)}{2r!}$$

Hence, by subtraction,

$$A_{2r}=\frac{(r+x-1)(r+x-2)\ldots(x-r)}{2r!}$$

$$\therefore \qquad u_x=u_0+x\Delta u_0+\frac{x(x-1)}{2!}b_0+\frac{(x+1)x(x-1)}{3!}\Delta^3u_{-1}+\ldots$$

Subdivision of an Interval.

$$(1+\delta)^m u_0=u_1=(1+\Delta)u_0$$

Hence,

$$\delta=(1+\Delta)^{\frac{1}{m}}-1=\frac{\Delta}{m}-\frac{m-1}{2}\frac{\Delta^2}{m^2}+\frac{2m^2-3m+1}{6}\frac{\Delta^3}{m^3}-\ldots$$

$$\delta^2=\left\{(1+\Delta)^{\frac{1}{m}}-1\right\}^2=\frac{\Delta^2}{m^2}\left\{1-\frac{m-1}{2}\frac{\Delta}{m}+\frac{2m^2-3m+1}{6}\frac{\Delta^2}{m^2}-\ldots\right\}^2$$

$$=\frac{\Delta^2}{m^2}\left\{1-(m-1)\frac{\Delta}{m}+\frac{11m^2-18m+7}{12}\frac{\Delta^2}{m^2}-\ldots\right\}$$

and generally,

$$\delta^n=\frac{\Delta^n}{m^n}\left\{1-\frac{m-1}{2}\frac{\Delta}{m}+\frac{2m^2-3m+1}{6}\frac{\Delta^2}{m^2}+\ldots\right\}^n$$

To express u in terms of w.

$$w_0\begin{cases}u_0\\u_1\\u_2\\u_3\\u_4\end{cases}\qquad w_1\begin{cases}u_5\\u_6\\u_7\\u_8\\u_9\end{cases}\qquad \text{\&c.}$$

$$w_0=u_0+u_1+\ldots+u_4=\frac{(1+\delta)^5-1}{\delta}u_0$$

Also $\quad \Delta w_0=(u_5-u_0)+(u_6-u_1)+\ldots+(u_9-u_4)=\{(1+\delta)^5-1\}w$

$$\therefore \qquad (1+\delta)=(1+\Delta)^{\frac{1}{5}}$$

Hence $\quad u_x=(1+\delta)^x u_0=(1+\delta)^x\dfrac{\delta}{(1+\delta)^5-1}w_0$

$$=(1+\Delta)^{\frac{x}{5}}\frac{(1+\Delta)^{\frac{1}{5}}-1}{\Delta}w_0$$

e.g.,

$$u_7 = (1+\Delta)^{\frac{7}{5}} \left\{ \frac{1}{5} - \frac{2}{25}\Delta + \frac{6}{125}\Delta^2 + \ldots \right\} w_0$$

$$= (1+\Delta)^{\frac{2}{5}} \left\{ \frac{1}{5} - \frac{2}{25}\Delta + \frac{6}{125}\Delta^2 + \ldots \right\} w_1$$

$$= \frac{1}{5}\left\{ 1 + \frac{2}{5}\Delta - \frac{3}{25}\Delta^2 + \ldots \right\} \left\{ 1 - \frac{2}{5}\Delta + \frac{6}{25}\Delta^2 + \ldots \right\} w_1$$

$$= \frac{1}{5}w_1 - \frac{1}{125}\Delta^2 w_1 + \ldots$$

or, if third differences vanish,

$$\frac{1}{5}w_1 - \frac{1}{125}\Delta^2 w_0$$

LEGAL NOTES.

By William Charles Sharman, F.I.A., *Barrister-at-Law.*

Assignment of Expectancy. Subsequent Bankruptcy of Mortgagor.

In making an advance upon the security of an expectancy or *spes successionis*, one of the difficulties it has been necessary to consider is the risk of the mortgagor becoming bankrupt and obtaining his discharge before the expectancy has become an interest. By Sections 37 and 38 of the Bankruptcy Act, 1883—now Sections 30 and 31 of the Bankruptcy Act, 1914—the effect of bankruptcy and of the order of discharge therein (subject to any conditions made by the order of discharge with respect to the after acquired property of the bankrupt) is to release the bankrupt from all liability under contracts creating debts provable in bankruptcy, and it has generally been accepted on the strength of the decision of the Court of Appeal in *Collyer* v. *Isaacs*, 19 Ch. D. 342, that in the case of a mortgage of an expectancy the bankruptcy of the mortgagor and order of discharge released the mortgagor from his covenant to assign. This decision has, however, been reversed in the recent case of *In re Lind* (1915), 2 Ch. 345, where the Court of Appeal upheld a decision of Warrington, J., to the effect that a mortgage by

assignment of an expectancy is not subject to any risk depending on the mortgagor becoming bankrupt and obtaining his discharge before the expectancy becomes an interest.

In deciding *In re Lind*, which was a mortgage of interests which the mortgagor might thereafter acquire in his mother's property in the event, which subsequently happened, of her dying intestate, the Court distinguished *Collyer* v. *Isaacs* on the ground that in the latter case, the assignment of the future chattels was construed, not as an actual assignment, but as a mere license to seize the future chattels.

The facts in *In re Lind* are as follows:

(1) By an indenture dated 21 February 1905, Hugh James Lawrence Lind mortgaged his interest expectant as prospective next of kin of his mother Florence Lind, a lunatic. (including property therein specifically mentioned), to the Norwich Union Life Insurance Society, to secure the repayment of £800 and interest.

(2) By an indenture dated 20 May 1908, Hugh J. L. Lind mortgaged his expectant interest (including property specifically mentioned therein) to Henry Lewis Arnold, subject, however, to the mortgage dated 21 February 1905.

(3) On 15 August 1908, Hugh J. L. Lind was adjudicated bankrupt. No proof in the bankruptcy was lodged by either of the defendants, the Norwich Union and Arnold, and by an order dated 12 October 1910, the said H. J. L. Lind obtained his discharge in bankruptcy.

(4) By an indenture dated 29 May 1911, between Hugh J. L. Lind of the one part, and the Industrials Finance Syndicate, Limited, the plaintiffs, of the other part, the said Hugh J. L. Lind in consideration of £260 sold and assigned his expectancy to the plaintiffs, subject to a trust thereby declared as to one moiety thereof in favour of himself.

(5) The said Florence Lind died intestate on 15 February 1914, and thereupon the said H. J. L. Lind became entitled (subject to the respective claims of the plaintiffs and the defendants, the Norwich Union and Henry Lewis Arnold) as one of the next of kin to one equal tenth part or share of and in her personal estates and effects. This share was estimated to amount to about £4,500, subject to the administrator's costs.

(6) By an indenture dated 6 March 1914 (endorsed on the indenture dated 29 May 1911) Hugh J. L. Lind, by way of further assurance, absolutely assigned to the plaintiffs all the share to

which he had become entitled, but subject to a trust as to one moiety in favour of himself.

(7) Letters of Administration to the estate of the said Florence Lind were granted to George McClintoch Lind, and as administrator he was made a party to the action.

The plaintiffs claimed a declaration that as assignees of Hugh J. L. Lind they were entitled by virtue of the indentures dated 29 May 1911 and 6 March 1914, respectively, to one equal tenth part or share of and in the personal estate of the said Florence Lind, deceased, subject as to one moiety of the said share to the trust declared in favour of Hugh J. L. Lind.

They claimed to be entitled to the fund on the ground that the assignments in favour of the Norwich Union and Henry Lewis Arnold were, in law, nothing but contracts to assign, creating debts provable in bankruptcy, and that by sections 37 and 38 of the Bankruptcy Act, 1883, the effect of the bankruptcy and of the order of discharge therein had been to release Hugh J. L. Lind from all liability under such contracts, and accordingly to render the assignments wholly ineffectual.

The Court of Appeal, consisting of Swinfen Eady, Phillimore, and Bankes, L.JJ., upheld the judgment of Warrington, J., that the Norwich Union and Henry Lewis Arnold were at the time of the bankruptcy entitled, not merely to the benefit of the personal obligation on the part of the mortgagor resulting in a claim for damages, but to a prospective interest in the distributive share in question, taking effect automatically on the death of Florence Lind; that the indentures of 21 February 1905 and 20 May 1908 were valid and effectual mortgages or charges of the share and interest of Hugh J. L. Lind as one of the next of kin of Florence Lind; and that the Norwich Union and Henry Lewis Arnold were both entitled to rank in priority to the plaintiffs.

In giving judgment, Swinfen Eady, L.J., said: "Is the "contention well founded that the rights of mortgagees of an "expectancy rest only on contract giving rise to a right of "proof if the assignor shall become bankrupt before the "expectancy vests in interest, but otherwise barred by the "bankruptcy and discharge of the bankrupt? In my opinion "the answer must be in the negative; the security of the "mortgagee remains in force and becomes effective whenever "the expectancy vests in interest."

Validity of an Equitable Assignment.

The validity of an equitable assignment of a *chose in action* was considered in the case of *German* v. *Yates and another*, reported 32 T.L.R. 52, and as the principles upon which the judgment was based would appear to be applicable to an assignment of a policy of assurance the case will doubtless be of interest.

The facts are as follows: Mrs. German some time before her death lent £100 to Sophia Yates and took from her an I.O.U. acknowledging receipt. Later she and Sophia Yates and Maria Yates met together and carried out the following transaction. Mrs. German said that she would like Maria Yates to have the benefit of the debt, to whom she asked Sophia Yates to pay the £100 when due. Sophia Yates agreed to do so, and Mrs. German thereupon tore up the old I.O.U. addressed by Sophia Yates to her, and Sophia Yates wrote out and handed to Maria Yates a new I.O.U. for £100 payable to the latter. On Mrs. German's death, the plaintiff, her widower and administrator, asked Sophia Yates to repay the £100 which he knew she had had from his wife. Sophia Yates set up the defence that the debt had been assigned to Maria Yates, and she refused to pay the plaintiff; and thereupon this action was brought, Sophia Yates and Maria Yates being made defendants.

Lush, J., in giving judgment for the defendants, said that in view of all the evidence he had come to the conclusion that the defendant's story was correct. That being the position, was there a good equitable assignment of the £100? For the plaintiff it was contended that equity would not assist a person who claimed under an imperfect gift, and would not complete an imperfect gift by turning the donor into a trustee for the donee (*Richards* v. *Delbridge*, 11 Eq. Cas. 15). That would be true if they were dealing with a chattel or with *choses in action*, such as shares in a company, which the law required to be transferred in some particular way.

Here the subject matter was a simple *chose in action* which could be transferred so as to give the transferee a right to sue under the Judicature Act and could also be transferred by equitable assignment. No form of words was required for an equitable assignment; the only thing that was necessary was to make the meaning plain (*Brandts* v. *Dunlop* (1905), A.C. at p. 462; 21 T.L.R. 710). It was further contended for the plaintiff that since the Judicature Act the creditor could make

a good legal assignment under the Act, and that if he purported to make an equitable assignment without consideration he was trying to do what could be quite well done in another way, and the transaction failed. But he (Lush, J.) could not accept that view. *Brandt's* v. *Dunlop* (at p. 461) showed that the Act had not destroyed equitable assignments or impaired their efficiency in any way, and they still existed alongside of the new kind of assignment under section 25. Nor could he accept the view that the assignment here was invalid because it was incomplete; it was perfectly good and complete.

There remained the question whether the assignment was invalid for want of consideration. It might be that if a creditor ordered his debtor to pay the money to some other person he might revoke the order if it was given without consideration; but if the creditor died without revoking it, his executor could not ignore the transaction. He held, therefore, that the plaintiff's action failed.

There was, he added, a further answer to it; that, in fact, the transfer here was made for good consideration. Sophia Yates entered into an obligation to pay Maria Yates, and even if in fact the assignment was ineffective and her obligation was non-existent, yet if she and all the parties believed, as in the circumstances they were justified in doing, that it was a valid transaction, there was sufficient consideration to support an equitable assignment.

Sale of business of Insurance Company. Failure to transfer Statutory Deposit.

In the case of *United London & Scottish Insurance Co.* v. *Omnium Insurance Corporation*, 84 L.J. Ch. 777, the House of Lords affirmed the judgment of the Court of Appeal decreeing specific performance of an agreement for the sale of the business of an insurance company, notwithstanding that the vendor company was unable to make a clean transfer of the £20,000 statutory deposit.

The facts are, briefly, as follows: An agreement was made between the two companies on 14 June 1911 for the sale of the respondents' business other than that of life assurance to the appellants, inclusive of the two deposits of £20,000 each deposited by the vendor company with the Paymaster-General under the provisions of the Assurance Companies Act, 1909.

Immediately after the sale agreement had been ratified by

the shareholders of the respective companies, the appellants, as purchasers, began to deal with the assets and business of the respondents as the proprietors thereof, and, after the agreement had been sanctioned by the Court, took entire possession and control of the assets and business comprised in the sale agreement. Although the sale agreement included the employers' liability business of the respondents, yet the appellants, not having any statutory deposit in Court which would enable them to carry on business of that character, found themselves unable, without the assistance of the respondents, to carry on business of that class; but, being desirous of retaining the goodwill of that branch of the business which had been transferred to them, an agreement was entered into between the two companies, under which the respondents agreed to permit the appellants to issue employers' liability policies in the name of the respondents until such time as the appellants should themselves have a deposited fund which would entitle them to carry on such business in their own name. The agreement secured to the respondents a proper indemnity against all claims in respect of such policies.

The petition for transfer of the employers' liability deposit to the appellant company came on for hearing in December 1911, but, in view of the liabilities to assured persons still resting upon that fund at that date, and being in fact increased by the acts of the appellants in issuing further policies in the name of the respondents, the Board of Trade opposed the transfer of the fund, and accordingly, by consent of the parties, the petition stood over generally.

No deposit for the purposes of employers' liability business was ever made by the appellants, and in February 1912, when they became financially embarrassed to a serious degree, they abandoned their intention of carrying on that class of business. On 22 February 1912 the appellants forwarded to the respondents for settlement a claim upon a fire policy issued by the respondents, and from that date forward the appellants consistently refused to discharge any further claims or liabilities in respect of the business of the respondents which had been taken over by the appellants, and caused all such claims to be forwarded to the respondents for settlement, repudiating liability therefor.

In these circumstances the respondents commenced an action claiming a declaration that their liabilities and engagements

ought to be satisfied and discharged by the appellants so soon as they matured or became payable, and for specific performance of the sale agreement so far as it remained unperformed. The appellants replied that the respondents had failed to carry the agreement into effect inasmuch as they had failed to make over to the appellants the employers' liability deposit.

Sargant, J., decreed specific performance of the sale agreement, and his judgment was affirmed by the Court of Appeal. The defendant company appealed to the House of Lords, but the appeal was dismissed.

The Lord Chancellor (Viscount Haldane) in his judgment said: "It is said there was a duty on the part of the respondent "company, as they could not transfer the £20,000 deposit, to "provide another deposit to take its place. I can find no "contract or trace of any such obligation in the agreement "which was entered into on 14 June 1911. Nor do I see that "the respondent company failed to do anything which they "could have done in order to make possible the transfer of the "deposit. The contract was not to transfer £20,000, but the "transfer of this deposit of £20,000, which was in Court, subject "to the rules relating to the employers' liability business; and "I do not find that the respondent company made any default."

Are premiums returnable on a policy void for want of insurable interest?

The case of *Hughes* v. *Liverpool Victoria Legal Friendly Society*, 31 T.L.R. 635, deals with the question whether premiums paid under a policy of life assurance which is void for want of insurable interest, can be recovered back. The case is one of a numerous class which is constantly coming before the Courts, and it would not have been thought necessary to refer to it in these Notes but for the fact that Scrutton, J., in the course of giving judgment for the defendants used the following words: "I decline to hold that in the case of a policy prohibited by "statute under penalty, money paid under it can be recovered "back if the payment was induced by a fraudulent misrepre-"sentation."

In view of these words, which at first sight appear to conflict with the generally accepted opinion on the question, it has been suggested that a short reference to the law on the point might be useful.

The general rule with regard to money paid in respect of an

illegal contract which has been wholly or partly carried out, is that the amount paid cannot be recovered back, the exception being that a payment so made by a party not in *pari delicto* with the other is still recoverable though the contract has been carried out. This rule was followed in the cases of *Cunliffe* v. *British Workman's Assurance Co.*, and *Harse* v. *Pearl Life Assurance Co.*, mentioned by Mr. Barrand in his paper entitled, "Further Notes on Some Legal Aspects of Life Assurance Practice", *J.I.A.*, vol. xli. p. 109. In the first case it was held that the parties were not in *pari delicto* and the premiums were ordered to be returned, and in the second the Court of Appeal held that the parties must be taken to be in *pari delicto*, and the premiums were not recoverable.

The statement of Scrutton, J., taken alone, is apparently inconsistent with the general rule, but taken in conjunction with his remarks in the course of his judgment, and particularly his reference to the plaintiff as a professional gambler in the lives of others, the inconsistency disappears, and had the question been raised there is little doubt that the parties would have been held to have been in *pari delicto*. The decision of the Judge is quite in accord with this view, and I think his statement should be taken more in the nature of an *obiter dictum* than an expression of the general law on the question.

Married Women's Policies of Assurance (Scotland) Act, 1880.

Divorce of Wife after policy effected.

I am indebted to Mr. G. J. Lidstone for calling my attention to a case which raised an interesting point as to the rights of a wife in a policy of assurance on the life of her husband under the Married Women's Policies of Assurance (Scotland) Act, 1880, where the marriage was dissolved by divorce for desertion at the husband's instance. The case in question is that of *Wallace* v. *Wallace*, reported 1916, 1 Scots. Law Times 163. The pursuer was Hugh Hutchison Wallace, and he sued his former wife, Mrs. Alberta Jane Robertson or Wallace and the Scottish Temperance Life Assurance Company, Limited, Glasgow, for a declaration that she had forfeited all right in a policy for £500 issued by the company in 1896 on his life for her benefit as his wife under the provisions of the Statute referred to, and that he had full right now to transact with the defenders the said Assurance Company, and to grant a sufficient discharge to them for any sum payable to him under the policy as the surrender value thereof, or to deal with it otherwise as

he and they might think proper. The material clauses of the policy were as follows:

"Under the provisions of the Married Women's Policies of Assurance (Scotland) Act, 1880.

"Whereas Hugh Hutchison Wallace, 20, Carrick Park, Ayr, Laundry Proprietor, being desirous to effect an assurance on the life of himself for the benefit of his wife, Mistress Alberta Jane Robertson or Wallace, in terms of the above recited Act and in virtue of the provisions contained therein (hereinafter called the assured) to the extent of the sum of five hundred pounds sterling, with the Scottish Temperance Life Assurance Company, Limited. . . .

"Now this policy witnesseth that if the said Hugh Hutchison Wallace shall die before or upon the Twenty-Seventh day of March, One Thousand Eight Hundred and Ninety-Seven, or shall live beyond such day, and shall, on or before the Twenty-Seventh day of March in every year during the continuance of this assurance pay to the said company the premium of fifteen pounds, nineteen shillings and two pence, then the said company shall be subject and liable to pay the sum of five hundred pounds sterling to the said Mistress Alberta Jane Robertson or Wallace who is hereby nominated trustee in terms of the said recited Act or to her legal representative in trust for the purposes hereinbefore expressed, within one calendar month after the death of the said Hugh Hutchison Wallace shall have been certified and proved to the satisfaction of the Directors of the said company."

Mrs. Wallace did not appear, but the company defended, and maintained that they were only bound to recognise the wife who was nominated as trustee under the policy as the sole person authorised by the statute to give a discharge, and in any event, they were not safe to transact with the pursuer without the sanction of a decree by the Court.

The pursuer had paid the annual premium down to the present time, and by bonus additions the policy had now acquired a considerable value. The parties were divorced in March, 1915.

Lord Anderson, in giving judgment, held that the statute as gathered from its preamble was intended to benefit married women and children in Scotland, and that the language of the operative section was only consistent with a subsisting relationship of husband and wife, and that any trust for the wife necessarily lapsed upon the cessation of such relationship. In

these circumstances, his Lordship held that the pursuer had now right to the policy, and he granted decree against the insurance company accordingly, but as in his opinion, the insurance company had properly defended the action in their own protection seeing the pursuer was unable to give them the receipt of Mrs. Wallace and stated their difficulties against payment, the defenders were entitled to the safeguard of his decree before paying to the pursuer, and he, therefore, held the insurance company entitled to expenses against the pursuer.

Income Tax. Is an agreement as to method of assessment in future years binding?

In the case of the *Gresham Life Assurance Society (Limited)* v. *Attorney-General*, 32 T.L.R. 264; (1916), W. N. 48, the point was raised as to whether an agreement entered into between the assurance society and the Surveyor of Taxes as to the amount which should be accepted as the estimated annual profit of the society until the next quinquennial valuation was binding upon the Crown so as to prevent the Crown during the next five years from assessing the society on an income instead of a profit basis, whatever the state of the law might be. It was held that the Surveyor had no power to make any such agreement.

The facts are as follows: The plaintiff company was formed in 1848 and has since carried on the business of a life assurance company. A large proportion of the company's business was and is carried on outside the United Kingdom, and it was therefore necessary that large investments should be maintained abroad, such investments in some cases being required to be invested in the national funds of those foreign countries. Before 1911. the plaintiffs were assessed to and paid income tax on income in accordance with the decision in Gresham Life Assurance Company (Limited) *v.* Bishop (1902 A.C., 287). The usual quinquennial valuation up to 1910 was made of the assets and liabilities of the company, and. as a result, correspondence took place between the company and the Surveyor of Taxes, it being the intention of the Surveyor to change his practice and thenceforward to assess the company on profits instead of on income.

On 19 April 1912. the Surveyor of Taxes wrote to the company enclosing an amended computation shewing the annual profit of £38,983, and stating that to be the liability for the year 1911-12 and the four succeeding years, and asking if the company agreed to the figure. The reply was in the affirmative.

The company paid its income tax for 1911-12 on the profit basis and for the two succeeding years they were assessed by the Surveyor at the same figures without being called on to make any return.

By the Finance Act, 1914, Section 5, all income arising from securities, stocks, shares or rents in any place outside the United Kingdom owned by any person or company resident in the United Kingdom is rendered liable to British income tax whether such tax is remitted to the United Kingdom or not. Accordingly, on October 14, 1914, the plaintiffs were asked to furnish information to the taxing authority so that it might decide whether it should revert to its former practice and tax the company on its income instead of on its profits.

The question arose whether the letters constitute a valid agreement between the plaintiffs and the Crown that the agreed figure of £38,983 should be accepted as the estimated annual profit of the company until the next quinquennial valuation, and that the Crown would not, during the next five years, whatever the state of the law might be, assess the company on an income instead of a profit basis.

Astbury, J., in the course of his judgment said :

"It is clear on the authorities that in assessing a society of "this kind there is an option to tax under Case 1 or Case 4, or 5, "Schedule D, or both, provided that the subject is not taxed "twice in respect of the same money.

"Three questions have been argued before me, namely, "(1) Could such an agreement as that alleged be entered into "with the Crown? (2) Had the Surveyor any authority to "make such an agreement? (3) Did the Surveyor purport to "make any such agreement? The plaintiffs say that this agree-"ment was made and acted on for five years. I see no evidence "of that. It is true that in 1912 an arrangement was made, "and that the four succeeding years are referred to. In the "years 1912-13 and 1913-14 the authorities in fact assessed "the plaintiffs on the agreed figures of their profits as shewn "by the past quinquennial valuation. But there is no evidence "that in so doing the authorities were acting or purporting to "act on any agreement which would prevent them, if they "thought fit, from reverting to another basis of assessment. "The income-tax is a yearly tax.

"The Solicitor-General says that in 1911 the taxing "authorities had no power to make any contract with the

"plaintiffs either as to the assessment to be made in subsequent "years, or as to the basis of making any such assessment. As "each yearly Act is passed the authorities have an option to "tax either on an income or a profit basis, and I think that in "each year that option should be exercised in a way most "favourable to the Crown. The duties and powers of the "Surveyor are defined in the Acts and he cannot go outside "them. I, therefore, find that the statutes give no power "either to the Board or the Surveyor in one year to tie the "hands of the authorities for four succeeding years. There "will be judgment for the Crown."

Courts (Emergency Powers) Act, 1914.
Foreclosure.

The case of *In re Farnol Eades Irvine & Co., Limited, Carpenter v. The Company* (1915) 1 Ch. 22, is of considerable importance to mortgagees, and in view of the large number of mortgages held by life assurance companies, will probably be of interest to their officials who deal with this class of investment.

The case in question was a debenture holder's action in which the plaintiff on behalf of herself and all other debenture holders claimed (1) a declaration of charge upon all the undertaking, assets, and effects of the company; (2) all necessary and proper accounts and enquiries; (3) payment of the amount secured; (4) foreclosure or sale; and (5) the appointment of a receiver and manager.

The writ was issued on 2 October 1914, and the plaintiff now moved for the appointment of a receiver and manager. The plaintiff had made no application to the Court under the Courts (Emergency Powers) Act 1914, (1) for liberty to bring or proceed with the action, and the defendants took the objection that in the absence of such an application the motion could not be entertained. It was held that a debenture holder can issue a writ of summons claiming the usual relief in a debenture holder's action and move for the appointment of a receiver and manager without any application to the Court under the Courts (Emergency Powers) Act, 1914.

Warrington, J., in deciding that the defendants' objection failed, made some observations on the word "foreclose" as used in section 1, subsection 1 (b) of the Courts (Emergency Powers) Act, 1914. He said: "Now I return to the word

"'foreclose.' What does that mean? Foreclosure as a "thing which can be done by a person has no meaning. Fore-"closure is done by the order of the Court, not by any "person. In the strict legal sense it is nothing more than the "destruction of the equity of redemption which has previously "existed. What, then, is meant in this clause by the pro-"hibition against foreclosure? I am not prepared to say what "it does mean, but I am prepared to say that in my opinion it "does not mean to prevent a person from issuing a writ of "summons for foreclosure. All that the Court does in such an "action is to direct an account of what is due on the security, "and that if it is not paid within a certain time, the "equity of redemption shall be foreclosed. But even that is "not foreclosure absolute, to obtain which the mortgagee has "got to get a further order. Now looking at the general "intention of the Act, what possible object can there be in "preventing a man from taking the preliminary proceedings "as the result of which he may obtain a judgment for foreclosure "from the Court. I see none whatever. It seems to me that, "whatever may be the meaning of the expression 'foreclose' "in this Act, it does not mean that a man may not put himself "in a position to obtain an order of foreclosure from the Court. "In my opinion, therefore, the Act does not prevent the issue "of a writ or originating summons for foreclosure or the issue "of a writ of summons in a debenture holder's action for the "realisation of the security. As to the appointment of a receiver, "if a receiver is appointed, possession is taken by the Court "by means of its own officer. There is no entry into possession "by any person by way of execution. The Court has power "to refuse the appointment of a receiver or to give any special "directions when and subject to what conditions the receiver "shall take possession. The control is left entirely to the Court, "and there is nothing in the Act to prevent the Court from taking "possession by the appointment of a receiver in a proper case. "The Court knows if the appointment of a receiver is necessary "in order to preserve the property and takes possession for the "benefit of all parties. In my opinion, therefore, the objection "taken by the defendants to the present motion fails, and I "make the usual order for the appointment of a receiver and "manager."

Statutory Deposit.
Application for payment of dividends.

In the case of *In re New York Life Assurance Company*, a petition for payment of dividends on the deposit paid into Court under the Life Assurance Companies' Act, 1909, the question was raised whether such an application could be made by summons or only by petition.

In the Annual Practice (1916), page 1000, it is stated that a petition is the only mode of making an application for this purpose, and it was so held in *In re Royal Exchange Assurance* (1910), W. N. 211.

Sargant, J., however, in making the order asked for, said that in his opinion a summons would be a perfectly proper mode of application for payment of the dividends.

The case is reported (1915) W. N. 376.

Income Tax.
Finance (No. 2) Act, 1915

The Income Tax Act 1853 granted relief from income tax in respect of premiums on insurance policies, subject, amongst other conditions, to the limitation that exemption could not be claimed in respect of premiums exceeding in amount one sixth of the total income chargeable. Variations in income due to the war have adversely affected many policyholders, and it is therefore provided by the Finance (No. 2) Act 1915, Section 26, that in calculating the relief that can be claimed the income shall be taken to be that for the year ended 5 April 1914, where such income is greater than that for the year under consideration.

The section applies to the tax year ending 5 April 1915, to the current tax year, and to any future tax year which may include a period during which the present war continues.

Excess Profits Duty.
Finance (No. 2) Act, 1915.

A considerable part of the Finance (No. 2) Act, 1915, is concerned with the subject of Excess Profits duty. This duty is specially charged in respect of excess profits earned during the war period. It is not proposed to deal generally with the duty in these notes, but attention may be drawn to Clause 8 of the Fourth Schedule of the Act, which is as follows:

In estimating the profits no account shall be taken of income received from investments except in the case of life assurance

businesses and businesses where the principal business consists of the making of investments. Where account is taken of any such income—

(*a*) Any variation in the value of any of those investments which appears to the Commissioners of Inland Revenue not to be due to a variation in profits shall also be taken into account; and

(*b*) Where the income has been derived from profits in respect of which any payment or repayment of excess profits duty has been made under this Act, such deduction or addition shall be made in computing the profits as will make proper allowance for that payment or repayment of duty.

Friendly Society Valuations.

The attention of members of the profession may be drawn to the following extract from the recently issued Report of the Chief Registrar of Friendly Societies for the year ending 31 December 1914:

During the year a new form of Valuation abstract has been issued to Societies and Branches for use in all Valuations made as at 31 December 1914, and subsequently. Further and more detailed particulars are required on various actuarial matters connected with the Valuation, and fuller information is asked for on the important subject of investments and depreciation of securities. Section 28 of the Friendly Societies Act requires a Friendly Society to "cause its assets and liabilities to be valued," and the Valuer must, therefore, take proper account of the substantial value of the Assets. He cannot divest himself of this responsibility for giving due effect to depreciation, because the Society has failed to do so in its statement of funds: and the new form is designed to ensure that the question will be brought prominently to his notice, and that full information will be supplied by him as to the action which he has taken. It may be useful to remind valuers that the recognition of depreciation of funds in some cases will result in an increase in the realised rate of interest. In ascertaining the realised rate the Valuer should therefore, as far as possible, base his calculations on the amount of funds as affected by the depreciation to which, in his judgment, he considers it necessary to give effect; and to the rate so obtained due weight should be given in framing the valuation assumptions as to future interest earnings.

In the report for 1912 attention was drawn to the unsatisfactory character of many of the valuations which had been submitted, some of which were affected by serious errors of principle, while others were found to contain arithmetical errors, sometimes of

considerable extent. The same features continue to present themselves, and much time and labour continue to be expended in a systematic supervision of valuations which ought to be wholly unnecessary and which can only be justified by regard for the consequences, to the members of the societies concerned, which might follow from an unsuspecting dependence upon the reliability of certain of the valuations received.

The following two cases, which have recently come under observation, may be cited :

At the end of 1914 a scheme under section 72 of the National Insurance Act was submitted by a Branch of a certain Order, accompanied by a valuation report, but both were returned for correction. In September 1915 a complete valuation under section 28 of the Friendly Societies Act by the same Valuer was received, which reproduced the figures of the valuation which had previously been rejected. A month later a revised scheme under section 72 of the National Insurance Act was submitted accompanied by an amended valuation, also by the same Valuer, in which the value of the liability in respect of sickness benefit was more than doubled and a deficiency of 100*l*. converted into one of 2,300*l*. The previous error repeated in the statutory valuation was in no way explained, and there was nothing in the correspondence to show that the Valuer was conscious of any responsibility to those whom, but for the scrutiny made by the Registrar, he would have so grievously misled.

In another case the comparison with the Annual Return revealed a serious error in the amount of funds entered in the Valuation Balance Sheet of a District Funeral Fund. According to the Annual Return the Total Funds, after deducting amount standing to credit of surplus accumulation Fund, consisted of :

	£	*s.*	*d.*
District Funeral Fund - -	24,371	4	10
Other Funds - - - -	2,050	19	$0\frac{1}{2}$
Total - -	£26,422	3	$10\frac{1}{2}$

In the Valuation Balance Sheet the Valuer had entered on the credit side opposite "Total Funds" the amount of the Benefit Funds (24,371*l*.) and on the Debit side was an entry "Balances of Subsidiary Funds 2,008*l*." The Net Benefit Funds were thus understated by 2,008*l*., and the effect was to reduce the surplus by this amount. Since a District Funeral Fund is merely a re-insurance Fund any surplus shown in its valuation falls to be apportioned among the branches interested. In the present case the number of such branches was about 30 and the effect of the error in the valuation of the District Funeral Fund was consequently to upset the valuation balance sheets, surpluses being unduly minimised and deficiencies exaggerated, of the whole of these branches.

Some degree of improvement is noticed since the subject was systematically taken in hand, by this office, in 1912, and there are encouraging signs that societies begin to attach greater importance

than formerly to the possession of recognized actuarial qualifications by those who essay to take upon themselves the duty of valuing. There is, however, still room for much improvement and the new form of valuation abstract described above has been designed with the object of securing from valuers such increased precision of statement as to the methods followed by them as will enable the Registrar to be satisfied in each case, without undue labour, that the Valuer has not evaded or overlooked any point which ought to be fully before him, and that he has avoided certain forms of error into which particular valuers have been prone to fall. It follows that for the new form to be productive of its full intended benefit it must be regarded seriously and with an absence of the perfunctoriness, which has been a marked and, for the examiner, an unreasonably trying feature of many of the valuation returns which have been received hitherto. If the new form should fail to achieve its purpose the Registrar will apparently have no option but to resort to the action foreshadowed in the Report for 1912 and to include in the Annual Report the full particulars of those valuations which contain errors of such a nature as to indicate that they have not received proper care and attention.

ACTUARIAL NOTE.

A formula for the Rate of Interest in an Annuity-Certain.

By J. F. Steffensen, D.Phil.

THE literature on the subject of annuities-certain does not indicate a formula of practical utility giving, without reference to tables and with uniform and satisfactory approximation, the rate of interest in an annuity-certain.

We leave it to the reader to judge whether the following new formula possesses some advantage over existing ones:

$$\delta = \frac{4\sqrt{\lambda} - (\lambda + 3)}{a + \frac{1}{2}} \qquad . \quad . \quad . \quad (1)$$

where

$$\lambda = \frac{12a + 6}{2n + 1} + 3 \ . \qquad . \quad . \quad . \quad (2)$$

In this formula, n means the duration, a the present value of the annuity

$$a_{\overline{n}|} = \frac{1 - v^n}{i} .$$

The yearly rate, if required, may be found from the expansion

$$i=\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}+\ldots$$

Besides simple multiplications and divisions our formula requires only the extraction of one square-root, which can be done without the aid of tables. If logarithms are used, λ should be written in the form

$$\lambda=6\frac{a+0\cdot5}{n+0\cdot5}+3.$$

(1) may also be written

$$\delta=\frac{(3-\sqrt{\lambda})(\sqrt{\lambda}-1)}{a+\frac{1}{2}}.$$

An estimate of the error involved in the formula may be formed by means of the following table, showing the approximation to the interest per cent. for specified values of i and n.

n	$100i=2$	4	6	8
10	1·995	3·989	5·983	7·976
20	1·999	3·998	5·999	8·002
30	2·000	4·001	6·008	8·019
40	2·000	4·005	6·016	...
50	2·001	4·008	...	...
60	2·001	4·012	...	...
70	2·002	...	...	...
80	2·003	...	...	...
90	2·004	...	...	...
100	2·004	...	...	...

In the cases which occur in ordinary practice, or, roughly speaking, for values of ni situated in the interval $\frac{1}{2}<ni<2$, the error does not exceed about $0{\cdot}002i$ and is usually smaller. Also outside this interval the approximation is generally very fair, and the formula can hardly ever completely break down, as will appear from the method by which it was obtained.

This method is very simple. Considering first a continuous annuity

$$\bar{a}_{\overline{n}|}=\frac{1-v^n}{\delta} \quad . \quad . \quad . \quad . \quad (3)$$

this equation may be written in the form

$$\frac{\bar{a}_{\overline{n}|}}{n} = \int_0^1 v^{nt}dt \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The value of the integral may, with very fair approximation, be calculated by Simpson's formula

$$\int_0^1 f(t)\,dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$$

The result is

$$\frac{\bar{a}_{\overline{n}|}}{n} = \frac{1}{6}(1 + 4v^{\frac{n}{2}} + v^n) \quad . \quad . \quad . \quad . \quad (5)$$

whence
$$v^n = \left(\sqrt{6\frac{\bar{a}_{\overline{n}|}}{n} + 3} - 2\right)^2 . \quad . \quad . \quad . \quad . \quad (6)$$

From (6) we might find i by logarithms; but logarithms may be avoided and at the same time a more accurate result be obtained by writing, from (3),

$$\delta = \frac{1 - v^n}{\bar{a}_{\overline{n}|}},$$

consequently we have approximately

$$\delta = \frac{1 - \left(\sqrt{6\frac{\bar{a}_{\overline{n}|}}{n} + 3} - 2\right)^2}{\bar{a}_{\overline{n}|}} \quad . \quad . \quad . \quad (7)$$

As, with good approximation,

$$a_{\overline{n}|} + \frac{1}{2} = \bar{a}_{\overline{n+\frac{1}{2}}|} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

it is clear that from (7) we obtain a formula for the ordinary annuity by replacing n by $n + \frac{1}{2}$ and $\bar{a}$ by $a + \frac{1}{2}$. The result is the formula

$$\delta = \frac{1 - \left(\sqrt{\frac{12a + 6}{2n + 1} + 3} - 2\right)^2}{a + \frac{1}{2}} \quad . \quad . \quad . \quad (9)$$

which, introducing the notation (2), is more conveniently written in the form (1).

Similar formulas exist, of course, for the *amount* of an annuity-certain, $\bar{s}_{\overline{n}|}$ and $s_{\overline{n}|}$. We need not go through the calculations

again, as a formula for the amount can always be derived from a formula for the present value by changing n into $-n$ and a into $-s$. In this way we obtain, for instance, from (8)

$$s - \frac{1}{2} = \bar{s}_{n-\frac{1}{2}} \qquad . \quad . \quad . \quad . \quad (10)$$

and from (9)

$$\delta = \frac{\left(\sqrt{\frac{12s-6}{2n-1}+3}-2\right)^2-1}{s-\frac{1}{2}} \qquad . \quad . \quad . \quad (11)$$

or, written in two equations,

$$\left.\begin{aligned} \delta &= \frac{\mu+3-4\sqrt{\mu}}{s-\frac{1}{2}} \\ \mu &= \frac{12s-6}{2n-1}+3 \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (12)$$

Other interest problems may, with approximation, be treated in a similar manner.

If, in (6), we put $\bar{a} = \frac{1-v^n}{\delta}$ and solve with respect to v^n, we obtain, writing for abbreviation $\omega = \frac{6}{n\delta}$,

$$v^n = \frac{\omega^2+7-4\sqrt{\omega^2+3}}{(\omega+1)^2} \qquad . \quad . \quad . \quad (13)$$

the value of which expression may also be calculated without the aid of tables, remembering that $\delta = i - \frac{i^2}{2} + \frac{i^3}{3} - \ldots$ or $\frac{1}{\delta} = \frac{i}{1} + \frac{1}{2} - \frac{i}{12} + \ldots$ If, for example, we take $i = \cdot 04$, $n = 30$, we find $v^n = \cdot 30809$, the correct value being $\cdot 30832$. From this approximation we obtain $a_{\overline{n}|} = \frac{1-\cdot 30809}{\cdot 04} = 17 \cdot 298$, the correct value being $17 \cdot 292$.

In conclusion we may say that the method employed in this paper for a somewhat special problem has a more general bearing and may, in principle, be applied to the approximate solution of equations of the form

$$K = \int_0^1 F(t) x^t dt \qquad . \quad . \quad . \quad . \quad (14)$$

where K is a constant, $F(t)$ a given function and x the quantity to be determined by the equation. In more complicated cases it is advisable to begin by satisfying oneself that Simpson's formula is applicable with safety, which may be done by taking account of the remainder form of the complete* formula, or

$$\int_0^1 f(t)dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \frac{f^{\text{IV}}(\xi)}{2880} \quad . \quad . \quad (15)$$

where $0 < \xi < 1$.

* See, for instance, Seliwanoff, Differenzenrechnung, p. 32.

ADDENDUM.

BEFORE sending this Note to the printer, the Editors of the *Journal* have made various interesting suggestions which I am glad to acknowledge.

Mr. Spencer points out that the other well-known quadrature-formula

$$\int_0^1 f(t)dt = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] \quad . \quad . \quad (16)$$

which in the case under consideration happens to be a perfect cube, gives another very similar expression for δ, namely,

$$\delta = \frac{1 - \left(2\sqrt[3]{\dfrac{a + \frac{1}{2}}{n + \frac{1}{2}}} - 1\right)^3}{a + \dfrac{1}{2}} \quad . \quad . \quad . \quad (17)$$

Mr. Todhunter, on the other hand, remarks that the introduction of δ may be avoided altogether by writing

$$a_{\overline{n}|} = \frac{n}{6}\left[1 + 4v^{\frac{n}{2}} + v^n\right] - \frac{1}{2}(1 - v^n) \quad . \quad . \quad . \quad (18)$$

solving the quadratic and finding the rate directly from $i = \dfrac{1 - v^n}{a}$. The result is

$$i = \frac{4n\sqrt{3n^2 + 9 + 6a(n+3)} - 6a(n+3) - 6n(n-1)}{a(n+3)^2} \quad . \quad . \quad (19)$$

Of these, Mr. Spencer's formula produces, for the durations usually occurring in practice, results which are more nearly

accurate than mine. For the shorter durations the results are practically the same, but for $n=\infty$ Mr. Spencer's formula ceases to give a good approximation, while from (1) we obtain, then,

$$\delta=\frac{4\sqrt{3}-6}{a+\frac{1}{2}}=\frac{\cdot 93}{a+\frac{1}{2}}$$

approximately.

Mr. Todhunter's formula is superior for the shorter durations, the results obtained for ordinary durations being much the same as mine. For $n=\infty$, (19) gives $i=\frac{\cdot 93}{a}$ approximately. The complication of (19) in comparison with the two other formulas is more apparent than real, as the transition from δ to i is avoided.

J. F. S.

[The foregoing Note deals with a problem which, although not of practical importance, has become classical, and Dr. Steffensen's ingenious contribution to its solution will, we think, be read with interest. The relative accuracy of the several formulas is not of course so material as the essential idea of using the method of quadrature, but it may be noted that the quadratic $1+v^{\frac{n}{4}}+v^{\frac{n}{2}}=3\sqrt{\bar{a}/n}$, which may be obtained by combining the quadrature formula of the fourth degree with Simpson, seems to give a slightly better result than the formulas suggested in the Note. This formula and (16) are best applied by writing $\bar{a}=a+\frac{1}{2}(1-v^n)$. For the comparatively small term $\frac{1}{2}(1-v^n)$ we may substitute in the one case $(1-v^{\frac{n}{2}})\sqrt{a/n}$ and in the other $\frac{3}{2}(1-v^{\frac{n}{3}})(a/n)^{\frac{2}{3}}$, leading to

$$1+v^{\frac{n}{4}}+v^{\frac{n}{2}}=3\sqrt{a/n}+\frac{3}{2n}(1-v^{\frac{n}{2}})$$

and

$$1+v^{\frac{n}{3}}=2\sqrt[3]{a/n}+\frac{1}{n}(1-v^{\frac{n}{3}})$$

Both these formulas give good results. The accuracy of the several approximations depends, to a material extent, on taking $\delta=(1-v^n)/\bar{a}$, or $i=(1-v^n)/a$, instead of obtaining the result directly by logs from the value of $v^{\frac{n}{2}}$, $v^{\frac{n}{3}}$, or $v^{\frac{n}{4}}$ given by the formulas.—EDS. *J.I.A.*]

REVIEWS.

The Mortality Experience of 17 Swedish Life Companies.

(Communicated by Dr. J. F. Steffensen.)

THE volume before us, whose full title we state below,* is to some extent a sequel to the volume published in 1905 by 19 Scandinavian and Finnish Companies, comprising, as regards Sweden, 64,717 male and 4,257 female lives with respectively 6,107 and 202 deaths. As this experience was considered too small for founding select tables upon, the Swedish companies decided to supplement it with their later experience, rejecting, however, at the same time the period of observation before January 1st, 1895. The end of the period of observation was December 31st, 1906.

The number of cards† on which the new experience is founded, thus became 191,970 (males) and 19,369 (females); the number of deaths respectively 10,086 and 446.

The investigation was based on three different "statistical units": I—the medical examination; II—the person insured; III—the sum insured. The first of these was, however, considered the principal one. According to this (the Austrian) method a person is considered an additional new risk every time he successfully passes another medical examination for insurance. It follows that Method I is particularly suited to select statistics; also it has enabled the Swedish companies, with little extra trouble, to deal separately with the experience of each company, only combining them for publication. It was ultimately found that the difference in the rates according to Methods I and II is of no practical consequence.

The experience comprises only medically examined, healthy ("normal") lives, and the statistics for males and females have been kept separate throughout. As regards males, separate tables are given for "whole-life" (in Sweden payable at 90) and endowment assurances. No distinction is made between "with-profits" and "without-profits" experiences, as policies of the latter type are not in use in the Scandinavian countries.

The result is eleven sections, of which the following synopsis, useful on account of its explanation of the symbols heading each table, is given in the book and translated here for the convenience of English readers:

I. *Unit: Medical Examination.*

Symbol.	(1) Select Tables for the whole experience:
${}^{s}\mathrm{M}^{h}_{u}$	(*a*) males.
${}^{s}\mathrm{F}^{h}_{u}$	(*b*) females.

* Undersökning af dödligheten enligt erfarenheten hos sjutton svenska lifförsäkringsbolag. Läkareundersökta normala risker 1/I 1895–31/12 1906. Stockholm, 1915.

† Not necessarily "lives"; *see* below.

(2) Aggregate Tables for the whole experience :

$^{s}M_a^h$ (*a*) males.

$^{s}F_a^h$ (*b*) females.

(3) Select Tables for sections of the experience (males only) :

$^{s}M_{[a]}^{l}$ (*a*) whole life (to age 90).

$^{s}M_{[a]}^{bl}$ (*b*) endowment assurance.

(4) Aggregate Tables for sections of the experience (males only) :

$^{s}M_a^l$ (*a*) whole life (to age 90).

$^{s}M_a^{bl}$ (*b*) endowment assurance.

II. *Unit : Person.*

$^{p}M_{[a]}^h$ (1) Select Tables for the whole experience (males only) :

$^{p}M_a^h$ (2) Aggregate do. do. do.

III. *Unit : Sum Insured.*

$^{f}M_a^h$ Aggregate Tables for the whole experience (males only).

The volume contains a detailed account of the method followed in collecting, sorting and counting the data, which were entered, by the companies themselves, on cards of a type prescribed by the Committee. Fractional exposures were treated with exceptional and perhaps exaggerated care, calculating them in months, using for simplification the curtate duration in months. We may here warn the reader against the formulas (1) and (2) on p. 23 for the time of exposure ; it appears, however, from p. 26, that the correct process has been used in the actual calculation.

The transition from exposures and deaths to mortality-functions requires a few, very simple theoretical considerations which, however, are rendered less accessible to the ordinary reader by attaching new and even variable meanings to certain acknowledged international symbols, such as d_x and C_x.

In the notation of the volume, let $\epsilon(x)dx$ be the number of persons under observation in the interval from age x to age $x+dx$, and let d_x be the number of deaths in the interval from x to $x+1$:

then $$d_x = \int_x^{x+1} \epsilon(x)\mu(x)dx$$

where $\mu(x)$ stands for μ_x : and the "central death rate" C_x is found by the formula

$$C_x = \frac{d_x}{\int_x^{x+1} \epsilon(x)dx}$$

It is shown that, with considerable approximation, $C_x = \mu(x+\frac{1}{2})$. Similarly, for select data, the observations produce a function $C_{x,t}$, which is approximately equal to $\mu(x, t+\frac{1}{2})$, that is $\mu_{[x]-t+\frac{1}{2}}$. In this case, of course, the number of persons under observation is a function of *two* variables, $\epsilon(x, t)$, and *double* integrals take the place of single ones. Let $\epsilon(u, v)du dv$ be the number of persons whose age at entry lies between u and $u+du$ and who are under observation from v to $v+dv$ years after entry; then $C_{x,t}$ is expressed as a fraction whose numerator is the corresponding number of deaths

$$d(x, t) = \int_t^{t+1} dv \int_{x-(v-t)}^{x+1-(v-t)} \epsilon(u, v)\mu(u, v)du$$

and whose denominator is the same integral for $\mu(u, v) = 1$. These integrals are capable of a simple geometrical interpretation which the space at our disposal does not allow us to give.

Where the unit was the sum insured the observed mortality-function was, according to the very incomplete explanation in the volume,

$$q_x = \frac{d_x}{b_x + \frac{1}{2}[i_x - (u_x - d_x)]}$$

where

i_x = sum insured under observation in the year,

u_x = sum insured lapsed in the year for other reason than death.

d_x = sum insured lapsed in the year on account of death.

$b_x = b_{x-1} + i_{x-1} - u_{x-1}$.

The graduation of the experience was performed by Professor Ivar Fredholm. Five different methods were tried :

(1.) It was at first attempted to represent the mortality of each separate year of assurance by a Makeham curve, but the results did not look promising enough for this project to be carried out.

(2.) The graphical method was then resorted to, applying it not directly to the rate or force of mortality, but to the function $D_{[x]-t}$ at the rate of interest $3\frac{1}{2}$ per-cent, and assuming that the effect of selection need not be traced further than 20 years after entry. After performing a first graduation of this sort, the operation was repeated on the *differences* of the graduated values and continued until about four significant figures were secured in the result. During this work it appeared that a more accurate result might be obtained by the method described under (3), and the graphical method was, therefore, abandoned.

(3). The graphical representation seemed to indicate that the difference $\delta_t = D_{x+t} - D_{[x]+t}$ was, for not too large values of x, practically independent of this variable. It was, therefore, attempted to represent the observed values of $D_{[x]+t}$ by an expression of the form

$$D_{[x]+t} = D_{x+t} - \delta_t$$

where D_{x+t} has the form resulting from Makeham's formula and δ_t means an exponential term. The way in which the constants were determined is not perfectly explained; but it seems that at one stage of the operations the method of Least Squares was applied, assuming the weights to be equal. It was found, however, that the difficulties at the higher ages (above 75) could not be overcome, and so the method was given up.

(4.) The fourth method is related to that adopted by Sir G. F. Hardy in graduating the $O^{[M]}$ experience. The function graduated seems to be the same as that so far denoted by $C_{x.t}$ (the "central death rate"), but it is now* denoted by $C_{[x]+t}$, although this latter symbol has a fixed meaning in the international notation.† The first step was to graduate the "ultimate" table, which was done according to Makeham's formula, leaving out the first 10 years of assurance. Nothing is said about the way in which the constants were determined. The result is stated‡ as

$$C_x = 3{\cdot}560 + 1{\cdot}688q^{x-30}$$

where $q^{20} = 6$, from which it appears that the rates have been multiplied by 1000. For the first 10 years of assurance the formula was

$$C_{[x]+t} = A(t) + B(t)q^{x-20}$$

where $A(t)$ and $B(t)$ are polynomials of the third degree whose form and values are stated on p. 45 of the volume. The process employed in determining the constants was of a tentative character: it was, however, secured that there should be contact of the second order, where the curves join the ultimate curve. The success of this graduation can be tested by the table on p. 46, stating, in groups of 5 years, the sums of the ungraduated and graduated $3\frac{1}{2}$ per-cent annuity-values for the years of assurance

* P. 43.

† The symbol $D_{[x]+t}$ is, on the other hand, used in the ordinary sense.

‡ P. 45.

0 and 10. A small systematic deviation is traceable, as might be expected when the effect of selection is presumed to wear off after so short a period as 10 years.

(5.) The attempt to limit the period of selection to any definite term was, therefore, finally given up, and thus arose the distinctive feature of this mortality-investigation, namely, that the graduated functions are given for all durations and do not after a certain term merge into an ultimate table. The formula employed* was

$$C_{nx} = \alpha_n + \beta_n C_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the versatile symbol on the left now means the "central death-rate" for a person, aged x, who has been insured for n years. C_x is, for ages above 29, a Makeham-expression

$$C_x = a + bq^{x-30} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $a = \cdot 1925$, $b = \cdot 374236$, $\log q = \cdot 0382$; for $x < 30$, C_x was represented by a polynomial of the third degree, whose numerical values are given on p. 49. For α_n and β_n the following formulas hold

$$\left.\begin{aligned} \alpha_n &= \cdot 288 - \cdot 100\left(\frac{1}{68}\right)^{\frac{n}{5}} - \cdot 136\left(\frac{7}{10}\right)^{\frac{n}{5}} \\ \beta_n &= \cdot 444 - \cdot 081\left(\frac{1}{68}\right)^{\frac{n}{5}} - \cdot 192\left(\frac{7}{10}\right)^{\frac{n}{5}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

The way in which these numerical results were obtained is partly of a tentative character and is not fully explained. Starting with preliminary values for C_x, the values of α_n and β_n for the first ten years of assurance were determined by dividing the experience for each separate year of assurance into two sections and seeing that the totals of adjusted and unadjusted deaths within each section agree. For the following years of assurance the experience was grouped in fives. In this way the preliminary values of α_n, β_n, given on p. 48 were obtained, and a graduation of these resulted in the above definitive formulas for α_n, β_n. After this, the preliminary values for C_x were improved by assuming, in formula (1), the values (3) for α_n, β_n, and equating, for each value of x, the totals of adjusted and unadjusted deaths for all durations :

* Comparison with the tables shows that the unit of time implicitly assumed in (1)–(3) is the month, while that employed in the tables is the year.

the resulting table of C_x was, finally, graduated separately, with the result mentioned above.

The success of the whole graduation can be tested by the tables on p. 50–52, comparing, for suitable groups of ages at entry and durations, graduated and ungraduated deaths. The agreement is, on the whole, fairly satisfactory for ages at entry below 50 years. For higher ages at entry the graduated mortality is somewhat too low, which, however, may not be of very great practical importance.

It only remains to add that the aggregate tables were graduated by Makeham's formula, but applied to q_x, that is

$$q_x = a + bc^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the constants were determined by the King-Hardy method. The numerical values are:

Unit	10^5a	10^5b	c
Medical Examⁿ ...	218	11·8264	1·093693
Person	232	11·7815	1·093471
Sum Insured ...	229	13·9764	1·090345

Nothing is said about the success of these graduations.

As an appendix to the general account of the principles and methods adopted by the Committee, the volume contains a paper by Professor Fredholm on the choice of the basis on which premiums and reserves should be calculated. The space at our disposal does not allow a digression on this paper which is partly of a theoretical character,* and, in so far as it goes into the practical side of the question, deals chiefly with the question of the transition from the basis used hitherto by the Swedish companies to the new experience. Only we should like to say that the author's main object is to prove that if uniform or non-decreasing profits are to be maintained, and if the actual rate of interest on investments does not increase, then the margin allowed on the mortality must on no account exceed the margin on the rate of interest. According to this theory it is not sufficient to choose a lower rate of interest and a higher mortality than expected, and there may, according to the author, even be cases where a mortality-table, which at the higher ages exhibits a lower mortality than actual experience (such as the 17 English companies' which is much used in Sweden) is preferable.

We admit that it is hard for us to accept this last consequence

* The author himself expresses on p. 66 a certain doubt as to the practical applicability of his theory.

of the author's view on the subject, and that, rather than tolerate a mortality-table exhibiting a too low mortality we would suffer the inconvenience of decreasing profits on the individual policy (not on the aggregate, provided the company is old enough to have reached stability) and remedy this defect in other ways, if necessary by forming a special fund for maintaining the profits.

The introduction (p. 1–78) is followed by tables (p. 1*–432*), and we thus find united in one volume observations (ungraduated and graduated) as well as commutation-tables. For the convenience of foreign readers all the headings are accompanied by a translation into French.

The tables on p. 3*–139* contain the ungraduated select experience (months under observation, deaths, "central death-rates") for the various sections of the experience. The ungraduated aggregate experience is given on p. 140*–150*. On p. 151*–170* we find the graduated "central death-rates" (males only) for unit medical examination.

The greater part of the remainder of the volume (p. 171*–419*) is devoted to tables of commutation, of premiums, of annuity-values and premium-reserves deduced from these graduated rates. Nothing is said about the process employed for the calculation, which is the more regrettable, as $C_{x,t}$ is only an approximation. These tables are necessarily very elaborate as selection is traced for every duration (as far as year of assurance 74, although observations are not found beyond duration 51). The tables contain, among other valuable information, the values of $D_{[x]+n}$, $N_{[x]+n}$ and $\mathbf{a}_{[x]+n}$ at the rates of interest $3\frac{1}{2}$, 4, and $4\frac{1}{2}$ per-cent—at the last rate of interest, however, only for every fifth age at entry.

Finally, we find on p. 420*–428* aggregate commutation-tables at $3\frac{1}{2}$ per-cent for all the three units: medical examination, person, sum insured.

We reprint extracts of some of these tables, namely, for the select experience, specimens of the functions, "Central death-rate", $\mathbf{a}_{[x]+t}$, ${}_tV_{[x]}$ and $P_{[x]}$; further, as regards the aggregate experience, q_x and $\mathbf{a}_x$, the rate of interest being $3\frac{1}{2}$ per-cent.

It will be seen that the premiums are very much like those resulting from the mortality of the general Swedish male population. The mortality is low, as it is throughout the Scandinavian countries, and the select mortality presents a minimum shortly before age 30, as is also found in the mortality of the general population. This feature has disappeared in the aggregate table, doubtless only because in this case Makeham's formula was carried down to age 15. The aggregate table resembles the Danish $D^{M(5)}$ Table, but the mortality is a little higher throughout. A glance at the specimen-values of $\mathbf{a}_{[x]+t}$ shows that an analogy to the fact disclosed by the late Sir G. F. Hardy† with respect to the $O^{[M]}$ expectations of life, namely, that in certain cases, $e_{[x]+t} > e_{[x+n]+t-n}$, does at least not exist as regards the graduated Swedish annuity-values.

† "Principles and Methods, etc.", p. 147.

${}^{s}M^{h}_{n}$ *"Central death-rate" (approx.* $= 1000\mu_{[x]+t+\frac{1}{2}}$).

$[x]$	$t=0$	1	2	5	10	20	30	40	50	60	70
15	1·22	2·44	3·47	5·90	6·16	6·76	11·32	21·89	47·22	108·23	255·33
20	2·62	4·28	5·10	5·38	5·04	8·29	15·23	31·57	70·86	165·55	...
25	2·34	3·45	3·86	4·39	5·90	10·67	21·30	46·61	107·55	254·54	...
30	1·79	3·08	3·83	5·16	7·25	14·36	30·73	69·94	164·52	...	...
35	2·21	3·71	4·61	6·35	9·34	20·09	45·36	106·16	252·95	...	...
40	2·87	4·68	5·81	8·19	12·59	28·99	68·07	162·39	...	...	...
45	3·89	6·18	7·69	11·05	17·63	42·81	103·33	249·69	...	...	...
50	5·48	8·51	10·59	15·50	25·46	64·25	158·06	...	...	...	...
55	7·94	12·14	15·10	22·39	37·61	97·55	243·04	...	...	...	...

${}^{s}M^{h}_{n}$ $\mathbf{a}_{[x]+t}$ $3\frac{1}{2}$ *per-cent.*

$[x]$	$t=0$	1	2	5	10	20	30	40	50	60	70
15	22·51	22·29	22·09	21·56	20·78	18·50	15·61	12·27	8·82	5·69	2·96
20	21·84	21·63	21·44	20·91	19·83	17·17	14·02	10·55	7·20	4·34	...
25	21·16	20·91	20·68	19·96	18·67	15·71	12·34	8·86	5·70	2·96	...
30	20·21	19·92	19·64	18·82	17·37	14·13	10·62	7·23	4·35	...	...
35	19·09	18·76	18·46	17·53	15·93	12·47	8·92	5·73	2·97	...	...
40	17·84	17·48	17·14	16·12	14·38	10·76	7·30	4·37	...	...	...
45	16·47	16·07	15·70	14·60	12·74	9·07	5·80	2·98	...	...	...
50	14·99	14·56	14·16	12·98	11·06	7·44	4·43	...	...	...	...
55	13·43	12·97	12·54	11·32	9·38	5·93	3·01	...	...	...	...

${}^{s}M^{h}_{n}$ $1000{}_{t}V_{[x]}$ $3\frac{1}{2}$ *per-cent.*

$[x]$	$t=0$	1	2	5	10	20	30	40	50	60	70
15	...	9·78	18·71	41·90	76·82	178·22	306·50	454·63	608·19	747·27	868·5
20	...	9·84	18·36	42·85	92·03	213·72	358·21	516·89	670·54	801·48	...
25	...	11·59	22·55	56·39	117·60	257·41	416·69	581·46	730·44	860·09	...
30	...	14·50	28·20	69·02	140·81	300·71	474·37	642·39	784·83	...	...
35	...	17·08	33·26	81·56	165·37	346·78	532·58	699·69	844·58	...	...
40	...	20·24	39·46	96·41	193·83	396·75	590·86	754·88	...	...	...
45	...	24·05	46·94	113·74	226·20	449,36	647·92	819·04	...	...	...
50	...	28·68	55·70	133·95	262·36	503·57	704·76	...	...	...	...
55	...	34·32	66·40	157·29	301·92	558·66	776·02	...	...	...	...

x	1000 P_x $3\frac{1}{2}$ *per-cent*			$^sM^h_a$	
	$^sM^h_a$	Swedish Population 1881-90	17 English Companies	1000 q_x	a_x $3\frac{1}{2}$ %
15	10·62	10·61	12·31	2·63	23·34
20	11·97	12·24	13·79	2·89	22·50
25	13·45	13·85	15·64	3·29	21·53
30	15·66	15·92	17·97	3·92	20·41
35	18·57	18·70	20·95	4·90	19·14
40	22·24	22·30	24·84	6·43	17·71
45	26·91	26·89	30·08	8·84	16·14
50	32·89	32·95	37·08	12·59	14·45
55	40·62	41·22	46·41	18·48	12·66
60	...	...	...	27·68	10·82
65	...	...	...	42·09	9·01
70	...	...	...	64·63	7·27
75	...	...	...	99·91	5·67
80	...	...	...	155·11	4·24

One very natural question presents itself to the reviewer on looking through this important publication. Are the somewhat limited facts really sufficient for justifying the formation of select tables for all durations? Unfortunately nothing is said in the volume about this all-important question, and the synopsis on p. 50–52 would seem to indicate the contrary. The fact that a graduation on this plan was successful and several other graduations failed is not conclusive in this respect. Only where the general nature of the law of mortality is well known, and provided that the formula expressing that law does not contain too many constants, may we by graduating even a meagre experience arrive at reliable results. It seems, therefore, that before placing entire confidence in Professor Fredholm's very interesting graduation it would be desirable to have his formula applied also to other and more extensive experiences.

Nevertheless, the publication is of great interest as it is, and although now and then the reader may feel a little neglected when he is struggling to retain the unstable meaning of certain symbols, or to supply missing links in the explanations, he is bound, on the whole, to acknowledge the skill and energy with which this considerable piece of work, partly on a scale which has deterred others, has been brought to a conclusion.

Mortality Laws and Statistics. By ROBERT HENDERSON.

(Pp. 111 + vi. \$1.25. New York: John Wiley & Sons, 1915).

THIS book is the latest of a series of mathematical monographs published in New York under the editorship of Mansfield Merriman and Robert S. Woodward, and it is significant that they have

included a book on mortality in the series among such purely mathematical subjects as hyperbolic functions, vector analysis and functions of a complex variable. Mr. Henderson begins his book with a short but interesting account of a dozen mortality tables, and uses this as an introduction to the mathematical treatment of life contingencies and the construction and graduation of mortality tables. The chapter on life contingencies avoids all work relating to functions involving interest, but by using joint expectations and the probabilities of survivorship the reader is introduced to laws of uniform seniority and the usual approximate work with which actuaries are familiar when a mathematical law of mortality is assumed. This part of the work is put together in an interesting way, and some refreshing remarks are found as, for example, when Mr. Henderson compares the distribution of lives according to duration with a frequency curve, and mentions that the mode corresponds with the age at death giving the most probable duration of life, the median with the *vie probable* and the mean with the expectation of life. He might possibly have added that the force of mortality has the same relation to the life curve as the well-known $(x+a)/(h+kx+lx^2)$ has to the Pearson-type of frequency curves.

The chapters on the construction and graduation of mortality tables contain much that is well worth re-reading, although we must confess that we found the part dealing with summation graduation formulæ the least interesting in the book.

We have noticed three little blemishes—Makeham's second modification of Gompertz' law of mortality is ascribed to Hardy, a graduation formula originally given by Kenchington is rediscovered and appropriated by the author, and in the opening paragraph of the chapter on graduation it is stated that in a binominal distribution the mean square deviation is equally likely to be positive or negative when n is large, but although defensible the phrase seems to require some reservation because the rate of mortality (q) is very small.

The book is not one for a beginner, but can be recommended to those who have already done some reading in the theoretical side of actuarial work; it is well arranged, is written in an interesting way, and is a very welcome publication.

W. P. E.

Table of Compound Interest at $\frac{1}{8}$ per-cent and of Antilogarithms to 60 figures to base 1·00125. *By* J. J. STUCKEY, M.A., A.I.A.

(Pp. 116. £1. 1*s*. net. London: George Allen & Unwin, Ltd.)

THIS is a table of $(1{\cdot}00125)^n$ for integral values of n from -2000 to $+2000$, *i.e.*, of present values and amounts at $\frac{1}{8}$ per-cent, or of antilogs to base 1·00125. It was originally constructed to 18 figures with the object of showing "in one table the approximate value of 1 accumulated or discounted at any rate of interest to intervals

of $\frac{1}{8}$ per-cent for any period ", but was subsequently extended to 60 figures. The reasons for the extension are not expressly stated, but apparently the principal, if not the only, application of the extended table is to the calculation of logs and antilogs to a large number of places. It is with reference to this application, and as a possibly valuable addition to the select number of original tables designed to facilitate, or even to render possible, some very occasional academic calculation, or to serve as a basis for the construction of practical tables, that Mr. Stuckey's table seems to claim serious consideration. It challenges comparison, in fact, with such classical productions as Abraham Sharp's 61-figure table of the common logs of the primes up to 1100, Wolfram's 48-figure table of the Napierian logs of numbers up to 10009, and Peter Gray's 24-figure table of log $1{\cdot}0^r n$ where $r=3, 6, 9 \ldots 24$, and $n=1, 2, 3 \ldots 999$.

For purposes of comparison we may take the example worked out by Mr. Stuckey—log 343—disregarding the easy options offered by $343=7^3$ or $\cdot 343=\frac{1}{3}\times 1{\cdot}029$, since such options would not be generally available. In order then to make use of Sharp's or Wolfram's tables it would be necessary to express 343 by trial—or by reference to a factor-table—in some such form as $141\times 39\times 499\times(1-1/2744001)/8000$. In this expression the common log of $(1-1/2744001)$ would have to be found by evaluating the usual logarithmic series and multiplying the result by the modulus; the logs of all the remaining factors would be taken from the tables. To use Gray's tables we should develop 343 by the radix method in the form $10^2\times 3\times 1{\cdot}143\times 1{\cdot}0^3 291\times 1{\cdot}0^6 630$, etc. In using Mr. Stuckey's table the process to be adopted would depend to some extent upon the number of figures required in the result. A comparatively simple method could, no doubt, be used for 24 figures—the maximum obtainable by Gray's tables—but for the full 60 figures obtainable from Sharp Mr. Stuckey resolves $1/{\cdot}343$ into the factors

$$a^{855}\times 1{\cdot}0^2 1\times(1-{\cdot}0^5 1)\times 1{\cdot}0^2 a^{-47}\times 1{\cdot}0^5 a^{153}\times 1{\cdot}0^8 a^{-231}\div(1-\mathrm{R}),$$

where a is written for $1{\cdot}00125$ and R in the residual factor consists of ${\cdot}0^{12}$ followed by 48 figures. In the evaluation of the log of this expression the Napierian logs of $1{\cdot}0^2 1$ and $(1-{\cdot}0^5 1)$ are taken from an auxiliary table, those of $1{\cdot}0^2 a^{-47}$, etc., must be obtained by expansion (the power of a being given by the principal table) and that of $(1-\mathrm{R})$ must be calculated in the same way as $\log_e(1-1/2744001)$ but with less labour since R begins with 12 0's against the 7 of 1/2744001: the value of $\log_e a^{855}$ is obtained by multiplying $\log_e a$ (which is given to 63 figures) by 855, and finally the sum of all the Napierian logs must be multiplied by the 60-figure value of M.

It will be seen that Mr. Stuckey's method is a special kind of radix method adapted to his new table, but the process of determining the radices is not so systematic as the ordinary process and may involve a considerable amount of tentative work. Thus, in the

example under consideration, the natural course would be to take a^{857} as the first factor, since a^{-857} is the nearest entry to ·343, but this would be a false start; Mr. Stuckey tries a^{856} to begin with, but has to reject it on account of its leading to subsequent factors which cannot be evaluated without higher powers of a than are given in his table, and he eventually adopts a^{855} with an "auxiliary factor" $1{\cdot}0^{2}1$. In the case of this particular example it would certainly appear that the common log would be found with less work by Sharp's table—even if 343 could not be reduced to a better residual factor than 1/2744001. Whether this would be so in general it would be impossible to determine without further tests. Mr. Stuckey must presumably have carefully considered the relative merits of his new method and the old methods before undertaking the very great labour of constructing his table, and we should like to know what his conclusions were. His introduction makes no reference to the methods that have hitherto been used, and does not state on what grounds or for what purposes the new method is to be preferred.

To turn from the calculation of 60-figure logs to the ordinary operations of arithmetic entails a somewhat abrupt transition, but it is necessary because special attention is drawn in the introduction to the fact that the table can be used for multiplications, divisions, involution, etc. This is no doubt true, but an antilog table with a 4-figure argument and a 60-figure entry cannot be regarded as very suitable for ordinary use. It may be mentioned that the example given by Mr. Stuckey of the application of the table to involution—the evaluation of s^{420} (incorrectly printed S^{420}) where s is the $O^{M(5)}$ value of the Makeham constant—might have been specially chosen to emphasise the academic character of the table, for it is not clear that s^{420} could ever be required except for some such purpose as the calculation of the probability of 4 infants all attaining age 105.

The compound interest applications are of a more practical nature, although naturally they only require a comparatively small proportion of the 60 figures. Since $(1+i)^{t}=(1{\cdot}00125)^{kt}$, where $(1+i)=(1{\cdot}00125)^{k}$ and k and kt will in general be partly integral and partly fractional, and since the second power of ·00125 and (consequently) the second difference of $(1{\cdot}00125)^{n}$ are small, it follows that the amount or present value of 1 at any rate of interest, and for any value of $t<2000/k$, may be obtained from the table approximately by a double first difference interpolation, *i.e*, by interpolating first for k and then for $(1{\cdot}00125)^{kt}$. Similarly the rate of interest at which $a_{\overline{n}|}$—or a redeemable security—has a given value can be found by means of the table, with considerable accuracy, by first difference interpolation between trial rates corresponding to suitably selected powers of 1·00125. The examples given in the introduction could probably all be worked out with equal, or at any rate sufficient, accuracy, and with less trouble, by interest tables or a fairly extensive log table of the ordinary kind. Nevertheless, the table has instructive properties in connection with compound

interest problems, and it might perhaps with advantage have been published for actuarial purposes without the last 50 figures. A 10-figure pocket table of $(1{\cdot}00125)^n$, with an auxiliary table giving the values of k for all ordinary rates of interest, which could be used for solving any problem in compound interest and would even serve, if the worst came to the worst, as an inconvenient log table, would have been an attractive curiosity. But the table as it stands, in attempting to combine an interest ready-reckoner, a log table, and an original mathematical table of a highly specialized type, seems to us to aim at fulfilling three entirely incompatible purposes.

A Course in Interpolation and Numerical Integration for the Mathematical Laboratory (Edinburgh Mathematical Tracts, No. 2). By DAVID GIBB, M.A., B.Sc.

(Pp. 90. 3*s*. 6*d*. net. London: G. Bell & Sons, Ltd.)

WE have read this book with interest, and can cordially recommend it to any members of the profession who wish to extend their knowledge of the practical applications of the methods of Interpolation and Finite Integration. The author's point of view is somewhat different from that to which actuarial students are accustomed, but that is by no means a disadvantage; on the contrary, it tends to throw fresh light on the uses and limitations of the various formulas. Moreover, as the book is intended for general use in mathematical calculations, it includes some interesting formulas and methods which the actuary does not meet with in the course of his ordinary reading, such as Borda's and Haros's formulas for the calculation of the logs of the primes, the radix method of calculating logs, and Gauss's method of quadrature with unequal intervals. But the chief merit of the book consists in the numerous fully-worked-out arithmetical examples by which the application of the formulas is illustrated. We may mention specially the examples of the calculation of a log by the radix method, of the approximation to the roots of an equation by interpolation, and of the error due to the neglect of higher orders of differences.

The chapter on Interpolation is in some respects less satisfactory than the rest of the book. We do not see what purpose—other than that of establishing an algebraical identity—is served by deriving Newton's formula from Lagrange's; the formula has already been obtained in the preceding chapter for integral values of n, and the transition from integral to fractional values does not require another algebraical demonstration, but an assumption (namely, that the function may be approximately represented at the point where its interpolated value is required by the polynomial determined by the given values). The effect of obtaining the formula again by an entirely different method may well be to lead the student to suppose, in the absence of any express statement to the contrary, that it differs in some way from the previous formula. Then it is stated, in effect, on page 23, that if an interpolated value is required between the last two of a given series of terms, Newton's formula

will not give a good approximation, because it will involve the last two terms only; and again on page 25, that Stirling's formula "should not be used for determining the value of a function for values "of the argument near the beginning or the end of a series of obser- "vations, *as it would not then be possible to obtain a sufficiently high "order of differences*" (the italics are ours). These statements are very confusing, and the confusion is worse confounded in the present edition of the book by a misprint in the example on page 22—$f(3{\cdot}5)$ being twice printed for $f(1{\cdot}5)$—as the result of which Newton's and Stirling's formulas are represented as giving two entirely different values for the same quantity. The fact is, of course, that each of the formulas—forward, backward or central—will involve all the given terms provided it starts at the right place, namely, with the first, last, or middle term respectively, and that all the formulas will then give exactly the same result, subject to the last available difference being assumed to be constant for the purpose of obtaining the mean difference required in one or other (according as the number of given terms is odd or even) of the central formulas. The derivation of the central difference formulas from Newton's by algebracial transformation is simple, and no doubt sufficient for practical purposes, although not general, but Everett's elegant and convenient formula is not given. In the discussion of the problem of finding the value of x for which $f(x)$ has a given value, it might have been mentioned with advantage that, as an alternative to the method of inverse interpolation, the approximate value of x could be obtained by direct interpolation by Lagrange—$f(x)$ being taken for this purpose as the argument, and x as the function.

In the chapter on Numerical Integration the principal formulas of integration and approximate summation are clearly and simply obtained, and there is an instructive article on the case where the upper limit is not a tabulated value. The derivation of the quadrature formulas from Cotes's general theorem seems, however, unnecessarily complicated. These formulas can be readily obtained by assuming that $f(x)$ is a rational algebraical function of the second, third, or fourth degree as the case may be, integrating, and solving a linear simultaneous equation for the unknown coefficients, and this method brings out in each case the precise nature of the parabolic curve by which the curve of $f(x)$ is replaced. Weddle's Rule, which the author obtains by adding $\frac{1}{140}$th of the 6th central difference to the quadrature formula corresponding to the assumption that $f(x)$ is of the 6th degree, is, perhaps, better obtained by deducting $\frac{4}{5}$ths of the Three-Eighths (Simpson's Second) Rule, taken over two ranges, from $\frac{9}{5}$ths of Simpson's (First) Rule taken over three ranges. This is not only much simpler, but it has the important advantage of showing that Weddle's is not an independent Rule and that no new formulas can consequently be obtained by blending it with the other Rules.

Five-Figure Mathematical Tables. *Compiled by* E. CHAPPELL, B.Sc., &c.

(Pp. 320 + xvi. 5s. net. London and Edinburgh: W. & R. Chambers, Ltd.)

THIS collection of Tables contains a useful addition to the ordinary log tables in the form of a table of logs of logs (or logs of cologs), and the corresponding "anti" table—tables of lologs and illologs, to adopt the names which the compiler gives them. The main object of these new tables is to facilitate the processes of involution and evolution. The evaluation of c^n by ordinary log tables entails five entries if the multiplication of log c by n is performed by logs—one for log c, two for the logs of log (or colog) c and n (or $-n$), and two more for the successive antilogs. The new tables give the log of the log (or colog) and the anti ($\pm$ anti) log, and thus reduce the requisite number of entries to three. The process is much simpler in operation than in description—in fact the evaluation of a power becomes identical in form with that of a product except for the slight complication introduced by the change of sign when c is <1 or n is negative. Mr. Chappell legislates for this by using red ink for the lolog when c is <1 and for the argument of the illolog table when the sign has to be changed in taking the second antilog, and by such rules as "when performing a process of involution or evolution corresponding to a negative index merely change the colour of the lolog and then proceed exactly as for a positive index." The red ink and the rules are to us, we must confess, rather confusing. Apart from the facts that red ink is not always available, that when available it is often used to denote negative quantities, and that the attempt to print a book in two colours nearly always results (as in the present instance) in faults of alignment, it seems to us that, as each of the two new tables really consists of two entirely distinct tables, it might be better to recognize this by having *four* tables, *i.e.*, of lologs, locologs, illologs and illocologs—or whatever may be the appropriate name, according to Mr. Chappell's system of nomenclature, for the antilog of an antilog with its sign changed. The only rules would then be that the locolog must be used when c is <1—there would, of course, be no lologs for quantities <1—and that in taking out the final result the illocolog table must be used when c is <1 and n is positive or when c is >1 and n is negative. Incidentally it may be noticed that Mr. Chappell's evaluation of the form $a\,(b/c)^n$ by means of the lologs of a and b seems to entail an unnecessary number of entries; it would apparently be simpler in such a case to take the log of $\pm$ (log b − log c) in the usual way. A convenient application of the lolog table occurs in connection with Napierian logs, for since $\log_e c = \pm$ antilog (lolog or locolog c − log M), the Napierian log can be found, with the aid of the lolog table, by two entries.

In addition to the new tables the book includes tables of logs, cologs, antilogs, and trigonometrical functions and their logs—with a note on each page showing whether the proportional parts are additive or subtractive, and with an occasional warning, when

the differences are large and changing rapidly, that (first difference) interpolation may not be sufficiently accurate. The colog table seems a luxury rather than a necessity, but will be useful for taking out reciprocals. Mr. Chappell uses it, we notice, for divisions, and, in fact, whenever the ordinary log would have to be subtracted, but probably most computers would as soon subtract as add and, in any case, would sooner subtract than use two tables.* A paragraph in the preface is devoted to defending—from the position of an engineer or applied scientist—the limitation (as a rule to five) of the number of figures in the tabulated values, but we note that six figures are given in the antilog table. The antilogs used in conjunction with the five-figure logs may consequently produce in many cases the illusory accuracy against which Mr. Chappell enters a protest. This seems to be the case in the examples of the use of antilogs given in the introduction, for in none of the six examples is the result correct to the sixth figure, and in two out of the six the error affects the nearest fifth figure.

We cordially welcome Mr. Chappell's proposal to regularize the use of "log" instead of "logarithm," although it is not clear why the last three letters of the longer word should be condemned as "quite unpronounceable"—"rhythm" and "chasm" would apparently fall under the same condemnation. "Lolog", also, seems quite convenient. But we do not understand why the familiar and time-honoured "antilog" should be supplanted by an upstart and meaningless "illog."

* It is only fair to say that some authorities seem to consider a colog table indispensable. Prof. G. H. Bryan and Mr. T. G. Creak, for example, in an article entitled "The Bordered Antilogarithm Table" in the *Mathematical Gazette* for December 1915, state that "for operations involving both multiplication and division . . . logarithms of recipocals are constantly needed." They are of opinion that "either the table of antilogarithms or that of logarithms may be banished with advantage," and as a colog table cannot be combined with a log table, they propose to banish the latter and to use, for all purposes, an antilog table with two complementary arguments—the log of the entry being given at the left side and top of the table, and its colog at the right side and bottom, and both logs and cologs being found by inverse entry. A specimen table (an incidental feature of which is that five figures are given in the earlier part of the table where the logs are changing rapidly, and four figures in the latter part) is appended to the article.

An Elementary Manual of Statistics. *By* Arthur L. Bowley, Sc.D., F.S.S. *Second Edition.*

(Pp. 220. 5*s.* net. London: Macdonald & Evans.)

Although intended primarily for beginners Dr. Bowley's Elementary Manual is a book that may be read with advantage by every student of statistics. Perhaps its chief merit is that in Part II it introduces the reader to actual statistical data by giving some account of the official statistics relating to such subjects as population, trade, prices and employment. But Part I, which deals with the elements of

statistical method, contains useful information on various practical matters which might escape the student who approaches the subject exclusively from the mathematical side, *e.g.*, the use (or abuse) of percentages, the use of diagrams, the necessity for precision in the headings of tables, and the main points to be kept in view in considering published statistics. And, in view of the difficulty of obtaining examples for self-examination, the collection of exercises in Appendix I is a useful feature.

In the present edition practically no alteration has been made in Part I, but the official statistics in Part II have in many cases been brought up to a later date, with some consequential alterations in the author's commentary—particularly in the chapter on income and capital. Two new appendixes have been added. On the other hand, in the process of reprinting a line appears to have been accidentally dropped at the foot of p. 69 and half a line at the foot of p. 74.

An Inquiry into the Statistics of Deaths from Violence and Unnatural Causes in the United Kingdom. *By* WILLIAM A. BREND, M.D., &c.

(Pp. 80. 2*s.* 6*d.* net. London: Charles Griffin & Co., Ltd.)

DEATHS in the United Kingdom from other than natural causes form the subject of quite a number of official reports and returns, but only two of these—the Registrar-General's returns and the Home Office report on coroner's inquests—cover the whole of the ground, the others—such as the Home Office report on street accidents caused by vehicles and the Local Government Board return of deaths from starvation or accelerated by privation—dealing with special classes of deaths for various statutory or departmental purposes. The different returns are compiled on almost as many different bases as regards area, age-classification, causation, etc., each being a law to itself with reference to its special purpose, but without reference to the others, so that they cannot be systematically compared.

In certain cases, however, in which comparison is more or less possible, Dr. Brend has discovered material discrepancies, and it seems a reasonable inference that such discrepancies in fact exist throughout the statistics. Dr. Brend adduces evidence to show that some deaths which form the subject of inquests are never reported to the Registrar-General and are consequently omitted altogether from his returns. The remedy suggested for the system, or lack of system, under which this is possible is substantially more uniformity and co-ordination in compilation. Totals would then have to agree, and mistakes would be discovered and corrected.

Dr. Brend devotes Part III of his Inquiry to the discussion of deaths from overlying, burns, anæsthetics, and poisoning. The subjects are of importance and Dr. Brend's conclusions are of interest—in regard to the deaths of children from burning (the "flannellette question"), for example, he is of opinion that it is the style of clothing rather than the material of which it is made that is the

principal factor—but their relevance to the inquiry consists in the evidence they afford that the statistics are not sufficiently reliable to form a basis for definite action. The first thing is to get the statistics right, and this is an object that is of some interest both to actuaries and to accident insurance companies.

La Mortalité chez les neutres en temps de guerre. By Dr. L. Hersch, *Privat-docent in the University of Geneva*

(Pp. 36 + 4 diagrams. 1 fc. Paris: Giard & Brière; Geneva and Bâle: Georg & Co.)

This is a reprint of a remarkably interesting paper communicated by the author in April 1915 to the National Institute of Geneva. It is obvious that a great war must have a very considerable effect on the vital statistics of the countries directly or indirectly involved, over and above the deaths due to military or naval service, but Dr. Hersch has been unable to trace any previous attempt to investigate the subject systematically from a statistical point of view. The present paper touches upon the general question of civilian mortality (in addition to giving some observations on the effects of war on births and marriages), although it deals mainly—as indicated by the title—with the particular aspect of the subject in which small neutral countries, adjacent to the belligerent states, are chiefly interested. From a comparison of the deaths registered in 1870–1872 in the countries most closely affected by the Franco-German War with the deaths registered in the preceding and succeeding years, Dr. Hersch estimates that as against a total of about 141,000 deaths officially attributed to service in the War (including deaths in hospital from wounds or illness contracted on service) there was an excess of no fewer than 854,000 civilian deaths, or about six times the number of the deaths directly due to the War, namely, 500,000 in France, 229,000 in Prussia, 55,000 in Belgium 47,000 in Holland, and 23,000 in Switzerland. The records of 1865–7 give similar results on a smaller scale, as regards the War of 1866, for Austria, Belgium and Holland—Switzerland had not introduced federal registration, and the German figures are not given. From the detailed statistics according to age and sex, the extra mortality appears to have fallen mainly on the very young and the very old, and to have been slightly heavier among males than among females. In Belgium, nearly one-half of the excess deaths were of children under five years old; in Holland, over one-half; and in Switzerland, one-third. "On arrive ainsi à la conclusion, aussi poignante que "paradoxale, que les grands se font la guerre et que ce sont les tout "petits qui en sont les principales victimes."

It is possible that so far as the neutral countries are concerned the crude statistics may somewhat overstate the case. There must have been a heavy mortality among the refugees who migrated from the war zone into the adjoining neutral countries, and perhaps also among the large bodies of troops that crossed the frontiers into

Belgium and Switzerland, and it does not appear from the paper whether the deaths in such cases were registered in the countries in which they occurred, and if so, whether allowance has been made for them. But this does not affect the extra mortality in the aggregate.

Whether the experience of former wars is being repeated in the present war Dr. Hersch's paper affords little or no statistical evidence. The only 1914 statistics available when the paper was written were those of the Canton of Geneva, and these, although showing a heavy mortality, are obviously too limited and too much affected by the large foreign element in the population of the Canton for any reliance to be placed on them.

Dr. Hersch thinks that notwithstanding the progress in public health and in social providence the extra mortality will be found to be enormous. We shall not venture on prophecy, but shall await with much expectant interest the publication of the results of Dr. Hersch's further investigations.

CORRESPONDENCE.

LIMITED PAYMENT LIFE POLICIES. SPECIAL RESERVES FOR PROFITS AND EXPENSES.

To the Editors of the Journal of the Institute of Actuaries.

Sirs,—Mr. Todhunter's Hypothetical Office Premium Formula, given in his recent paper (*J. I. A.*, vol. xlix, p. 264) on the above subject, produces reserves equal to the net premium reserve for the sum assured plus the net premium reserve for a pure endowment equal to the amount of the special additional reserve required at the end of the premium period by the usual method. From the Table in the paper (p. 265), it will be seen that the full reserve in the illustration there given is 69·86 at the end of 20 years. The single premium at age 50, $O^{M(5)}$ $2\frac{1}{2}$ per-cent, is 61·49, and therefore, 8·37 is the amount of the pure endowment. The increase in reserve arising by valuing these pure endowments as endowment assurances would, perhaps, not be considered important in practice. If the premiums under limited payment policies are valued by the Z-method it might be found convenient to make a correction for the special reserve by, at the same time, scheduling an amount to be valued as an endowment assurance, which amount could be suitably adjusted to take account of any inadequacy in the limited payment premiums.

In the following table are given reserves calculated on this principle in comparison with the figures given in Column (6) of Mr. Todhunter's Table. There is also added, as a matter of interest, the reserves obtained by valuing 8·37 as a pure endowment on the experience basis $O^{(M)}$ 4 per-cent.

Limited Payment Policy Values.

n	By Hypothetical Office Premium Method (Mr. Todhunter)	Net Premium Reserve $+8{\cdot}37\,{}_nV_{30:\overline{20}}$ ($O^{M(5)}$ $2\frac{1}{2}$ %)	Net Premium Reserve ($O^{M(5)}$ $2\frac{1}{2}$ %) $+8{\cdot}37\,{}_nV_{30:\overline{20}\vert}^{\,1}$ ($O^{[M]}$ 4 %)
0	0·00	0·00	0·00
5	14·00	14·08	13·81
10	30·02	30·16	29·73
15	48·44	48·57	48·19
20	69·86	69·86	69·86

The difference in reserve would be greater in some cases than shown above, but if thought material a higher rate of interest might be adopted. Investigation would possibly show that the average age obtained for the valuation of the premiums, or a simple modification thereof, would be applicable to the endowment assurances. The method would perhaps be considered applicable to single premium policies, the usual extra reserve at the end of a fixed term of years being valued as an endowment assurance; or a portion only of the special reserve could be provided in this manner, the balance being provided for in the usual way.

Apparently it would be necessary to reschedule a correction factor at the end of the premium period, (in the case of single premiums at the end of the fixed term agreed upon), but possibly some convenient device could be found to overcome this. If not, there would be some compensation in the fact that a considerable proportion of policies cease before the end of the premium period, thus materially reducing the work.

Yours faithfully,

A. D. WATSON.

Insurance Department,
Ottawa,
9 *October* 1915.

[A practical method of valuation (for $n < t$), suggested by the foregoing letter, would be to take the net premium value and add n/t-ths of the additional loading reserve required at the end of the payment term. An alternative method—suggested by the writer of the letter with the object of reducing the valuation-strain in the early years incidental to taking n/t-ths of the additional loading reserve—would be to add to the net premium reserve the accumulated sinking fund for the additional loading reserve, *i.e.*, to take ${}_nV_{\overline{t}|}$-ths of that reserve instead of n/t-ths. Either of these modifications would answer the purpose of the hypothetical office premium method, that purpose being merely to avoid the inconvenience of the ordinary method of valuation—consistently

with making a fully adequate reserve—by distributing over the payment term the valuation-loss due to the commutation of the whole-life premium at a rate of interest higher than the valuation-rate. The first mentioned method, which has the advantage of being very simple in application, would give the following reserves for $n=5$, 10 and 15 in the case dealt with in the letter:—14·54; 30·82; 49·11.—EDS. *J.I.A.*]

INSTITUTE OF ACTUARIES.

THE SIR GEORGE HARDY MEMORIAL.

A SPECIAL Meeting of the Institute of Actuaries was held at Staple Inn Hall, Holborn, on Tuesday, 11 January 1916, for the purposes of receiving the dedication of the Memorial Fund which had been subscribed in commemoration of the late Sir George Hardy's life and work, and of witnessing the unveiling of a bust executed for the Institute by Mr. Gilbert Bayes.

The attendance—representative, so far as the circumstances of the times permitted, of all classes of members—bore witness to the respect and affection with which Sir George Hardy was regarded by the profession.

Several ladies occupied seats in the gallery, among those present being Lady Hardy.

The PRESIDENT, in opening the proceedings, said that they had met to do honour to a man for whom they had the greatest esteem and who was beloved by all of them. It was not his intention to make a long speech, because there were others present who knew Sir George Hardy better than he did, and who would be able to speak with fuller knowledge of his great abilities and of the great regard they all had for him. They were there that evening in a double capacity. The Council had decided that the Institute, as a corporate body, should possess in its Hall a permanent memorial of their dear friend, and those who knew him personally had decided, as private individuals, to do something to perpetuate his memory. He would call upon Sir Gerald Ryan to state what been done.

Sir GERALD RYAN said that the occasion was unique, for he believed that no similar gathering had taken place in the history of the Institute. In past times they had had to mourn the loss of distinguished sons of the Institute, men who had passed away in the fulness of years, leaving a fine record of work behind them. But even among them their dear friend, the late Sir George Hardy, stood alone. He was an exceptional man—exceptional in the rare gifts and endowments with which he was blessed; exceptional in the value and volume of his services; exceptional, too, alas, in the hour and circumstance of his death, for it was while his mental vigour was at its highest that the hand of death fell, all too soon, upon him, to their profound sorrow.

The two-fold recognition which they proposed to pay to his memory aimed at different objects and proceeded along different lines. By the Bust which the President would presently unveil they sought to testify to their desire to retain in their midst as a perpetual memento of a dear friend something of his personal traits and characteristics, something of his charm of feature, of his high intellectual cast of countenance. But by the Memorial Fund which they were establishing they endeavoured to perpetuate and to encourage those mental influences and scientific pursuits with which his remarkable career was associated; the one made a human appeal to them, the other an intellectual appeal.

He was unmoved by the cold philosophy of those who maintain that the highest merit needs no monument, that good work is indestructible and provides its own imperishable monument. He believed that the higher promptings of their nature were totally opposed to such a view. A national assembly without its busts of the great figures of history, a fine city without its mementoes of illustrious patriots and benefactors, would leave too much to memory and the imagination. In this particular case, the extraordinarily spontaneous reception given to their movement showed the wide conviction that some such memorial was highly appropriate and desirable in Hardy's case. For himself he thought they would be strangely lacking in generosity, appreciation and gratitude if they failed to take some such steps as these to do honour to his memory.

In the Dedication, which he would presently read, it would be found that the Committee on which he had had the high honour of serving as Chairman, and whose actions had been controlled by their admirable Secretary, Mr. Warner, had succeeded beyond expectations in its task. Assisted by generous donations from Lady Hardy, the partner of all Sir George's triumphs and trials, and other members of the family, and by contributions from members of their profession abroad, they had collected a fund of no less than £767. The wide-spread character of the reception which had been given to the appeal testified to the unusual popularity and esteem in which Hardy was held throughout the world, and when they considered the services which he had rendered to the profession he thought they would all agree that neither the bust, nor the memorial, nor both together, furnished an adequate return. Hardy was not only the best lecturer they had ever had: not only the most brilliant contributor to their proceedings; not only a dignified and distinguished President of the Institute. He was all that, and he was much more than that. To him, he thought, belonged modern actuarial science more than to any man of his time, and the extent of his influence upon their work could not yet be accurately gauged. In earlier days he (the speaker) had, for many years, a very close association with Hardy, and he was impressed immensely with his instantaneous comprehension of a subject, his intuitive perception of the essential difficulties of the problem, and his bold and dexterous manœuvring for a solution. In those respects he was the most gifted man he had ever met.

It was not only within their own body that Hardy had rendered noteworthy services. Many departments of the State called him in to advise them in connection with important Government affairs, and in the closing months of his life he had received from his Sovereign a great honour worthily won by him and enthusiastically welcomed by all his friends. As to his personal traits, one hesitated to speak. His charm of manner endeared him to everyone with whom he came in contact; and he had among the junior members of the profession a following which, in popularity and intensity, had probably never been equalled. To him all science seemed well within reach, and he was able to master any branch of it, and, indeed, to advance any branch of it, without that preliminary study which every average mortal had to undergo. In those respects indeed he was probably without an equal.

He would ask the President formally to accept the Memorial Fund, on behalf of the Institute, in the terms of the following Dedication—

"*To the President and Council of the Institute of Actuaries.*

"The Committee which was formed to raise a Fund in commemoration of the life and work of the late Sir George Hardy, K.C.B., having now brought its operations to completion, hands to the Institute of Actuaries as the result—

	£	*s.*	*d.*
$4\frac{1}{2}$ per-cent War Loan Stock ...	700	0	0
Cash	67	15	9
	£767	15	9

"The response to the appeal has been a thoroughly representative one, 137 contributions having been received, 125 of which are from Members of the Institute of Actuaries in Great Britain and the Colonies.

"The Fund has been raised in order that the annual income arising therefrom be applied for the assistance of actuarial education and research among the younger members of the profession, in such ways as may from time to time—in its administration by the Institute of Actuaries—be deemed most effective.

"It is, therefore, now handed over to the Institute in trust for these purposes; to be known as 'The G. F. Hardy Memorial Fund.'

"On behalf of the Committee,

(Signed) G. H. Ryan, *Chairman.*
S. G. Warner, *Secretary.*"

The Committee had now completed its allotted task, and their final wish was that the Memorial Fund might be a new element of strength to the Institute, and might perhaps in fulness of time produce another man of the high type and eminence of their friend Hardy. In any case it would give great encouragement to those who, without the assistance of this endowment, might find it difficult to devote their leisure and work to the purpose. He hoped, indeed, that many a

student in future would draw inspiration and would date his success in life from his having been what he might call a "Hardy scholar of his year."

When reflecting on the larger things in life he sometimes permitted himself to put a fanciful test to those who occupy high stations, and to wonder whether they were better than the positions they hold or whether the position overshadowed the individual. In professional life one might similarly ask whether a man had done more for his profession than his profession had done or could do for him. Those who survived such a test as that were very few, and they were the elect. Powerful minds, great versatility, and great powers of imagination were theirs; and he believed that Hardy would easily and irresistibly have won a place for himself in that select band, no matter where his lot in life had been cast. Fortunately for them, his guiding star directed that he should follow their important but, comparatively speaking, humble profession, and there the successes of his life were won. For all he did for them and for their position as a scientific body and for their aspirations they owed him unstinted admiration and lasting gratitude, and all of them of his generation would always hold his name in affectionate remembrance.

Mr. S. G. WARNER said that it might perhaps seem that their memorial to Sir George Hardy was not altogether adequate. In a deep sense, of course, no such memorial ever could be adequate. They need not hope to commemorate such a life and influence as his by anything in the nature of a material gift. The things were incommensurable. But apart from that, from the more limited standpoint, one would have been glad to see the response, good as it had been, greater still, and greater it would undoubtedly have been but for the times of stress and strain and unprecedented difficulty during which the appeal had to be made. Having regard to those times, they could not but be gratified by the widely representative character of the response, and they could not but feel that in essence their end had been achieved.

To some of them it was by no means a cause for entire regret that they should have been called to work like this at such a time. Surrounded as they were by tidings of conflict, destruction and carnage, which almost shook one's faith in the future of humanity, they had been recalled by this work to consider a life which was great and pure and noble, which was devoted to that pursuit of exact knowledge and study of natural law which know no distinction of race, and make for the binding of all mankind in one. It had been said, and well said, that in a very deep and true sense Sir George Hardy needed no memorial. His memorial was in his written work—so brilliant, so original, so lucid; and, far more than that, in what was behind his work, in his spoken word, in the abiding spell of his personality. Therefore he would like to think of their own enterprise as regards this Fund not perhaps altogether or primarily as one of commemoration, but rather as one of perpetuation, of endeavouring to carry on and continue something which they knew would have been on the lines of his dearest desires. If he were asked to sum up in

one word the secret of the charm of his life and personality, difficult as that task would be, and, in its perfect fulfilment, impossible, he should be inclined to choose, in its noblest and widest sense, the word "helpfulness." When they thought over their memories of him, that note constantly recurred. The patience, so kindly and unfailing, with any honest effort, however poor or limited; the guidance over rough places and through dark passages so constant and so sure; the generosity so ready to appreciate and rejoice in any success, however modest—these were the things that recurred to their minds. These things now, and happily for many years to come, were living memories. They influenced the spirit and the activities of those who still remain.

But the time would come when they would be forgotten, and their science would endure. It would have new votaries and new students; and in their science, as in all others, the pathway of the student and beginner became more difficult as years went by. Fresh accumulations of knowledge and research had to be absorbed; new methods had to be comprehended and mastered. It was because they felt this, and because they knew how strongly he whom they now commemorated felt it and endeavoured to meet that want, that they had tried to establish something which should be, from year to year, some source of help and encouragement not only to the present generation of students but to those yet to come. They had endeavoured by a certain elasticity in the principles on which it was based, to give it a form which would enable its particular application to be varied from time to time as circumstances changed; thus preventing the evil which sometimes came from too rigid a constitution—good enough at the beginning, but eventually an obstacle to usefulness. In this way they hoped to have achieved a work which would be lasting and beneficial. And they had secured this—that for all time, as long as their science and students of it endured, one great and illustrious name would be associated with thoughts and ideas of help and of encouragement, and that they had been enabled to join together in this effort and to link with it the name of so great a teacher, so great a guide, so great a friend, would by subscribers and Committee alike, be held in record as a high honour and treasured as an abiding satisfaction.

Mr. F. B. Wyatt desired to say a few words in expression of his great satisfaction that this memorial had been created. During almost the last three years of Sir George's life he had been associated with him more closely perhaps than any other Member of the Institute. During those three years Sir George Hardy had been engaged on what was perhaps the greatest and most useful work of his life. He was called upon to advise on the great national insurance scheme, and in that connection he (the speaker) had worked with him nearly every day and for many hours a day. There was no difficulty that he could not solve; nothing was too complex for him. They had often to work under great pressure. He, himself, used to feel somewhat downhearted and anxious, but he was always reassured by the calmness and the ingenuity and the wonderful skill that their friend, Sir

George Hardy, brought to bear on these difficult subjects. He did not think that the scheme would ever have come to fruition without the assistance of such a man as that. He hoped that he would be pardoned for making this personal reference, and he would like to add that during those three years he learned a great deal from Sir George Hardy. He learned, above all, not only to admire his great intellectuality and intelligence, but to appreciate his wonderful nature, which never made an enemy.

Sir A. W. Watson thought that it was personality, not less than intellectual power, that attracted them so strongly in Sir George Hardy. In this respect his professional brethren were not alone. In connection with his latest and perhaps his greatest piece of work he had seen Hardy exploring difficulties, both practical and technical, with statesmen and with departmental officials: and his influence was such that when the time for taking decisions arrived his opinions and his advice were found to have prevailed to a much greater extent than his professional capacity alone would explain. He could only think that that was largely due to his unobtrusive but compelling personality.

There was no intellectual arrogance about Hardy. Absolute unaffectedness and simplicity seemed to be the note of his every action. He would give an attentive and patient hearing to the views of anyone who was associated with him on the work in hand. It did not matter that the man was quite a youngster, with all his career to achieve. If he had anything to contribute to the common stock, Hardy was always willing to listen. No one who worked with him could help feeling that Hardy was a real colleague and a faithful co-worker. Again, he adopted no attitude of mere superintendence and direction. In no way did he spare himself. Never did he allow a feeling to be engendered in the minds of other people that he was taking less than his full share of the common labour. His marvellous capacity enabled him to do in a few minutes what would take an ordinary man many hours. Notwithstanding that, the time that he devoted to the common task when work was at its heaviest was as great as that put in by those whose responsibilities were more immediate and might well be regarded as more exacting than his.

But, above all things, he was a very trusted guide to those who worked with him. They realized they had in him a man who had no other purpose than to serve faithfully those who had entrusted their interests to his hands, and one who, if any difficulty arose, would give a full and loyal support to his colleagues.

Mr. G. J. Lidstone said that the occasion was a sad but also a memorable one, and no one of them—no actuary throughout the whole world, who had had the opportunity of knowing either the man or his work—would willingly miss this opportunity of paying a tribute to the memory of Sir George Hardy. To all he was a great leader of actuarial thought; to many he was a dear personal friend. Previous speakers had dealt with his work in terms more eloquent than he (the speaker) could command, and he would not attempt

to add much to what they had said; indeed, it was difficult to speak at all without some repetition. Sir George Hardy's work was universally appreciated, and perhaps most highly so by those who were best able to judge of it. It was not too much to say, as Johnson said of Goldsmith, that he had left but little untouched, and had touched nothing that he did not adorn. His work had that enduring quality that is associated with genius, and it would always be a permanent monument to his memory, so that they were rather fulfilling a duty to themselves than doing honour to him in making a public memorial as they were doing to-day. Nothing could be more fitting than that that memorial should be connected with the advancement of the younger members and nothing would have been nearer to his own heart. If there was one thing more remarkable than his work it was perhaps his influence on the generation which succeeded him, and in that way his wonderful personality had woven itself into the very fibre of actuarial science. He spoke in the presence of many who, like himself, had the great privilege of being his pupils, and surely they must all feel that their indebtedness to him was incalculable.

It was nearly a quarter of a century ago that Sir George Hardy had urged him to go in for the final examination, which he had been rather inclined to postpone, and in doing so had probably changed the whole current of his life. It was still perfectly fresh in his memory how they had all looked forward to his lectures. He was a kindly and luminous lecturer, and in every way almost a perfect tutor. No trouble was too great for him to take, in regard to either the class as a whole or any individual pupil who wanted special light on any point. Even the blunders of a thinking pupil were often the means of his developing something for their guidance and instruction. When, later on, some of his pupils became his friends, they then almost lost their admiration for his work in their personal affection for the man. No branch of knowledge was foreign to him. He was able to meet experts on numbers of subjects on their own ground, and for those who were not experts it was not too much to say that to know him was a liberal education. His singularly magnetic personality allowed few, if any, who met him to escape its charm. In a very long friendship with him he did not remember his making any remark or expressing any view which even bordered remotely on the unkind or the uncharitable. He had a generous appreciation of other people's work, even when it was, as he must have known, far inferior to his own. He (the speaker) regarded Sir George Hardy's friendship as one of the great outstanding privileges of his life, and his memory would always be ineffaceable in their hearts.

The President said that on behalf of the Council of the Institute he gladly accepted the memorial which had been handed to him, and he was sure they would endeavour to administer it in the spirit in which the subscribers would wish it to be administered. He was sure they would allow him to add his tribute to the memory of their friend, especially as he would approach the matter from a somewhat different point of view to the previous speakers. Unfortunately for him, he had not enjoyed the privilege of knowing Sir George

Hardy intimately, as others had. His only opportunities of seeing him were at the meetings of the Council of the Institute and other official gatherings during the last 19 years of his life. What always struck him when he did meet Sir George Hardy was his extreme modesty, and the genial and kindly way in which he listened to whatever anyone had to say. He (Mr. Woods) was impressed by the strange contrast there was between such a man—a man of peace—and his times. He was born just after Sebastopol had fallen and before the declaration of peace, he entered official life just as the Franco-German War was finished, and he was called away from them while the first great battle of Ypres was being fought. He (the President) was sure they would all join with those who had already spoken in paying a tribute to the memory of their friend—a tribute not only to his intellectual qualities, but to his kindly disposition towards all of them. They all felt he was indeed a true friend. Shortly after his death, the Council heard that the bust which was being executed by Mr. Gilbert Bayes was still unfinished. They felt that the best thing they could do was to secure that bust for their Hall, and Mr. Bayes had very kindly completed it, and produced what he was sure they would all consider to be a work of art.

The President then formally uncovered the bust, which is of green bronze and is a faithful likeness of Sir George Hardy. It stands upon an oak base bearing Sir George's name and title in gold letters. It also records the years during which Sir George occupied the position of President of the Institute.

The Committee consisted of the following:

Sir Gerald H. Ryan, *Chairman.*

Mr. T. G. Ackland.
Mr. H. W. Andras.
Mr. W. P. Elderton.
Mr. D. C. Fraser.
Mr. C. D. Higham.
Mr. L. F. Hovil.
Mr. W. Hutton.
Mr. G. King.
Mr. A. Levine.
Mr. G. J. Lidstone.
Mr. G. Marks.
Mr. W. P. Phelps.
Mr. F. Schooling.
Mr. J. Spencer.
Mr. R. R. Tilt.
Mr. R. Todhunter.
Mr. H. M. Trouncer.
Sir A. W. Watson.
Mr. J. D. Watson.
Mr. W. J. H. Whittall.
Mr. E. Woods.
Mr. F. B. Wyatt.

Mr. S. G. Warner, *Treasurer.*

SUSPENSION OF SESSIONAL MEETINGS AND EXAMINATIONS OF THE INSTITUTE.

The following notice was issued to members at the beginning of the current session:

STAPLE INN HALL,
HOLBORN, W.C.

November 1915.

DEAR SIR,

Owing to the absence of so many members of the Institute on War Service, and to the great amount of work which has consequently fallen on those who remain, the Council have found it necessary to reconsider the question of holding the Ordinary General Meetings of the Institute during the current Session.

After full deliberation the Council have decided not to hold any Sessional Meetings until further notice, and consequently the meetings arranged for the 29th November, and subsequent dates, will not take place.

The Council have also resolved, in view of the special circumstances arising in connection with the War, to abandon the Annual Examinations arranged to be held in April, 1916.

We are,
Yours faithfully,

A. D. BESANT,
J. BURN,
Honorary Secretaries.

Obituary.

ERNEST HAROLD MARDEN GUMPRECHT, B.Sc., Probationer of the Institute, Private, London Rifle Brigade.

Killed in Action 3 *May* 1915.

Richard Clift Fippard, Fellow of the Institute, Captain, 14th West Yorkshire Regiment (attached Lancashire Fusiliers).

Killed in Action in June 1915.

John Bernard Evelyn Tombs, Probationer of the Institute, Lance-Corporal, 9th Battalion Middlesex Regiment.

Died from Dysentery 23 *September* 1915.

Hubert Charles Alfred Gravatt, Associate of the Institute, Private, Honourable Artillery Company.

Killed in Action 30 *September* 1915.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Central-Difference Interpolation Formulæ. By W. F. SHEPPARD, Sc.D., LL.M.

MR. D. C. FRASER has lately (*J.I.A.*, vol. l, p. 22) pointed out that there is need of a simple proof of the central-difference formulæ for Lagrangian interpolation.

I.—SHORT METHOD.

For the mathematical student I should imagine that the simplest way is to use the general Newtonian formula.

1. We first obtain the ordinary advancing-difference formula for $n+1$ u's at equal intervals

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2\,!}\Delta^2 u_0 + \ldots$$
$$+ \frac{x(x-1)\ldots(x-n+1)\Delta^n u_0}{n\,!}.$$

This is only required for the purpose of showing that the coefficient of x^n in u_x is $\Delta^n u_0/n\,!$.

2. Next consider the general formula for the case of unequal intervals. Let

$$u_a,\ u_b,\ u_c,\ \ldots\ u_f,\ u_g$$

be the values of u corresponding to

$$x=a,\ x=b,\ x=c,\ \ldots\ x=f,\ x=g;$$

these values of x not necessarily being in order of magnitude.

Let $F(x)$ be the formula for u_x when we use n u's $u_a, u_b, u_c, \ldots u_f$, and $\Phi(x)$ the formula when we use these and u_g. Then, if

$$\Phi(x) \equiv p_0 + p_1 x + p_2 x^2 + \ldots + p_n x^n,$$

the p's are given by $n+1$ equations

$$\left.\begin{array}{l} p_0 + p_1 a + p_2 a^2 + \ldots + p_n a^n = u_a \\ p_0 + p_1 b + p_2 b^2 + \ldots + p_n b^n = u_b \\ \quad \vdots \qquad \vdots \qquad \vdots \\ p_0 + p_1 f + p_2 f^2 + \ldots + p_n f^n = u_f \\ p_0 + p_1 g + p_2 g^2 + \ldots + p_n g^n = u_g \end{array}\right\}.$$

Let $\Phi(x) \equiv F(x) + \phi(x)$. Then $\phi(x)$ is a polynomial of degree n in x, and it must be $=0$ for $x = a, b, c, \ldots f$. It therefore contains $(x-a)(x-b)(x-c) \ldots (x-f)$ as a factor, and we may write it

$$\phi(x) = (x-a)(x-b)(x-c) \ldots (x-f)\{a, b, c, \ldots f, g\},$$

where $\{a, b, c, \ldots f, g\}$ is some expression which may involve $a, b, c, \ldots f, g$ and $u_a, u_b, u_c, \ldots u_f, u_g$, but does not involve x. Hence, commencing with u_a, and taking in the other u's successively, we have

$$\Phi(x) = u_a + (x-a)\{a, b\} + (x-a)(x-b)\{a, b, c\} + \ldots$$
$$\ldots + (x-a)(x-b) \ldots (x-f)\{a, b, c, \ldots f, g\}.$$

3. Here $\{a, b, c, \ldots f, g\}$ is the coefficient of x^n in $\Phi(x)$, and therefore is the value of p_n as given by the $n+1$ equations in § 2. But these equations can be arranged in any order, and we shall still get the same p_n. Hence the $a, b, c, \ldots f, g$ in $\{a, b, c, \ldots f, g\}$ can be arranged in any order without altering its value, *i.e.*

$$\{a, b, c, \ldots f, g\} = \{b, a, c, \ldots f, g\} = \{a, c, b, \ldots f, g\} = \text{\&c.}$$

4. Now suppose that $a, b, c, \ldots f, g$ are the same as $0, 1, 2, \ldots n$, but not necessarily in this order, so that the intervals are not necessarily equal. Then, by § 3,

$$\{a, b, c, \ldots f, g\} = \{0, 1, 2, \ldots n\}.$$

But this latter expression is the coefficient of x^n in the formula for u_x in terms of $u_0, u_1, u_2, \ldots u_n$; and, by § 1, this coefficient is $\Delta^n u_0 / n!$. Hence this is the value of $\{a, b, c, \ldots f, g\}$.

5. This gives us the general "zig-zag" formula for equal intervals; but we need not write this down, as we only require the central-difference formulæ. We take the x's in the order $0, 1, -1, 2, -2, 3, -3, \ldots$; and then we have

$$\begin{aligned}\Phi(x) &= u_0 + x\{0, 1\} + x(x-1)\{0, 1, -1\} + x(x-1)(x+1)\{0, 1, -1, 2\} \\ &\qquad + x(x-1)(x+1)(x-2)\{0, 1, -1, 2, -2\} + \ldots \\ &= u_0 + x\{0, 1\} + x(x-1)\{-1, 0, 1\} + x(x-1)(x+1)\{-1, 0, 1, 2\} \\ &\qquad + x(x-1)(x+1)(x-2)\{-2, -1, 0, 1, 2\} + \ldots \\ &= u_0 + x\Delta u_0 + x(x-1)\Delta^2 u_{-1}/2! + x(x-1)(x+1)\Delta^3 u_{-1}/3! \\ &\qquad + x(x-1)(x+1)(x-2)\Delta^4 u_{-2}/4! + \ldots\end{aligned}$$

From this the various formulæ are easily obtained.

II.—Detailed Treatment.

For the ordinary student, the following seems a fairly natural procedure.

6. We require some preliminary propositions:—

(*a*) $u_n = u_0 + (n, 1)\Delta u_0 + (n, 2)\Delta^2 u_0 + \ldots + (n, n)\Delta^n u_0$. This is easily proved by induction, or from the identity $E^n = (1+\Delta)^n$. Similarly

$$u_0 = u_n - (n, 1)\Delta u_{n-1} + (n, 2)\Delta^2 u_{n-2} - \ldots + (-)^n (n, n)\Delta^n u_0.$$

(*b*) If $\Delta^n u_0$ were zero, the last term in the above expression for u_n would be zero. Hence

$$u_n = v_n + \Delta^n u_0,$$

where v_n is the next term obtained by continuing the sequence $u_0, u_1, \ldots u_{n-1}$ forwards on the assumption that the $(n-1)$th difference is constant. Similarly

$$u_0 = v_0 + (-)^n \Delta^n u_0,$$

where v_0 is the next term obtained by continuing the sequence $u_1, u_2, \ldots u_n$ backwards on the assumption that the $(n-1)$th difference is constant.

(*c*) It is instructive to obtain (*b*) by a different method. Taking the first case, we can split up the sequence $u_0, u_1, \ldots u_{n-1}, u_n$ into two, viz.:

$$v_0, v_1, \ldots\ldots v_{n-1}, v_n$$
$$w_0, w_1, \ldots\ldots w_{n-1}, w_n$$

where $u_r = v_r + w_r$. Then we shall have

$$\Delta^n u_0 = \Delta^n v_0 + \Delta^n w_0.$$

Now take $v_0, v_1, \ldots v_{n-1}$ to be $u_0, u_1, \ldots u_{n-1}$, and v_n to be the next term obtained by continuing this sequence with constant $(n-1)$th difference. Then

$$\Delta^n v_0 = 0.$$

The w's will be

$$0, 0, \ldots 0, w_n,$$

and we shall have

$$\Delta^n w_0 = w_n.$$

Hence

$$\Delta^n u_0 = 0 + w_n = w_n;$$

and therefore

$$u_n = v_n + w_n = v_n + \Delta^n u_0.$$

7. The Lagrangian formula for interpolating the value of u_x by means of (say) the six values $u_1, u_2, \ldots u_6$ is obtained by assuming that the u's are the values, for $x = 1, 2, \ldots 6$, of a polynomial in x of degree 5; *i.e.* that $u_x = \mathrm{F}(x)$, where

$$\mathrm{F}(x) \equiv p_0 + p_1 x + p_2 x^2 + \ldots + p_5 x^5,$$

the p's being given by the condition that

$$\mathrm{F}(1) = u_1,\ \mathrm{F}(2) = u_2, \ldots \mathrm{F}(6) = u_6.$$

This condition gives 6 equations

$$p_0 + p_1 + p_2 + \ldots + p_5 \qquad = u_1,$$
$$p_0 + 2p_1 + 4p_2 + \ldots + 32p_5 = u_2,$$
$$\&c.,$$

which determine the p's as linear functions of the u's; and therefore, substituting in $\mathrm{F}(x)$, we can write the latter in the form

$$\mathrm{F}(x) = \mathrm{X}_1 u_1 + \mathrm{X}_2 u_2 + \ldots + \mathrm{X}_6 u_6,$$

where each X is a polynomial in x of degree not exceeding 5.

8. It is important to notice that—

(i) The interpolation-process consists in taking u_x to be equal to $F(x)$ as defined by the conditions that it is of the form $p_0+p_1x+\ldots$ and that $F(1)=u_1$, $F(2)=u_2, \ldots$ The determination of $F(x)$ itself from these conditions is a purely algebraical process and has nothing whatever to do with interpolation. It is this algebraical process that we are concerned with.

(ii) These conditions give one $F(x)$ only; and therefore, if we find a polynomial of degree not exceeding 5 which makes $F(1)=u_1$, $F(2)=u_2, \ldots F(6)=u_6$, we know that this is $F(x)$.

9. One form of $F(x)$—the ordinary Newtonian form—is obtained at once from § 6 (*a*) above. The equality

$$u_x=u_1+(x-1,\ 1)\Delta u_1+(x-1,\ 2)\Delta^2 u_1+\ldots+(x-1,\ 5)\Delta^5 u_1$$

holds for $x=1, 2, 3, 4, 5$, and 6; and it follows from § 8 (ii) that

$$F(x)=u_1+(x-1,\ 1)\Delta u_1+(x-1,\ 2)\Delta^2 u_1+\ldots+(x-1,\ 5)\Delta^5 u_1.$$

10. Here $F(x)$ is expressed in terms of a certain set of differences. (The word "difference" is here, and elsewhere, used for brevity to include the u's themselves, which are differences of order 0.) But $F(x)$ can also be expressed in terms of other sets of differences. Let us first write it in the form

$$F(x)=X_1u_1+X_2u_2+\ldots+X_6u_6.$$

Then we know that

$$\Delta^5 u_1=u_6-5u_5+10u_4-\ldots-u_1.$$

Hence we can remove either u_6 or u_1 from $F(x)$ by taking out a term $+X_6\Delta^5 u_1$ or $-X_1\Delta^5 u_1$. In the first case the portion left involves u_1, u_2, u_3, u_4, u_5; in the second case it involves u_2, u_3, u_4, u_5, u_6. Similarly in the first case a term in $\Delta^4 u_1$ can be taken out so as to remove either u_5 or u_1; and in the second case a term in $\Delta^4 u_2$ can be taken out so as to remove either u_6 or u_2. Proceeding in this way, we see that there are $2^5=32$ different expressions for $F(x)$ in terms of $\Delta^5 u, \Delta^4 u, \Delta^3 u, \Delta^2 u, \Delta u, u$; starting from $\Delta^5 u_1$ and proceeding always diagonally to one or other of the two differences which lie nearest in the system of differences (including the u's). The coefficients will in all cases be polynomials in x of degree not exceeding 5.

11. We have next to find what these coefficients are. To do this, we assume that we know $F(x)$ for the above case of 6 u's; and we examine what will happen if we introduce another u, which may be either u_7 or u_0.

(i) First, suppose it is u_7. Let the expression for interpolation from these 7 u's be $\Phi(x)$. Then $\Phi(x)$ is a polynomial of degree 6, given by the condition that

$$\Phi(1)=u_1,\ \Phi(2)=u_2,\ \ldots\ \Phi(6)=u_6,\ \Phi(7)=u_7.$$

Let us write

$$\Phi(x)\equiv F(x)+\phi(x).$$

Then, since

$$F(1)=u_1,\ F(2)=u_2,\ \ldots\ F(6)=u_6,$$

it follows that

$$\phi(1)=0,\ \phi(2)=0,\ \ldots\ \phi(6)=0.$$

Also, since $F(x)$ is of degree 5 in x, the 6th difference of the sequence $F(1)$, $F(2)$, ... $F(7)$ is zero; or, to put it differently, $F(7)$ is the next term in the sequence u_1, u_2, ... u_6 continued on the assumption that the 5th difference is constant. It follows, as in § 6 (b), that

$$\phi(7)=\Delta^6 u_1.$$

But $\Phi(x)$ is of degree 6, and $F(x)$ is of degree 5; and therefore $\phi(x)\equiv\Phi(x)-F(x)$ is of degree 6. Hence the problem is to determine a polynomial $\phi(x)$ of degree 6 which shall be $=0$ for $x=1, 2, \ldots 6$ and $=\Delta^6 u_1$ for $x=7$. This polynomial is obviously

$$\phi(x)=\frac{(x-1)(x-2)\ \ldots\ (x-6)}{6\,!}\Delta^6 u_1;$$

and therefore this is the term which has to be added to $F(x)$ in order to allow for the fact that we are taking u_7 into account.

(ii) Similarly for using u_0, u_1, u_2, ... u_6 in place of u_1, u_2, ... u_6 we should find that the term to be added is

$$\frac{(x-1)(x-2)\ \ldots\ (x-6)}{6\,!}\Delta^6 u_0.$$

12. This gives us the law of formation of the successive terms. Suppose, for instance, that we start with u_4. The first value of $F(x)$, of course, is the "constant" u_4. The next expression is found by adding $(x-4)\Delta u_4$ or $(x-4)\Delta u_3$ according as the next u to be taken into account is u_5 or u_3. In the first of these two cases the u's involved are u_4 and u_5, and therefore the next value of $F(x)$ is

$$u_4+(x-4)\Delta u_4+\frac{(x-4)(x-5)}{1\,.\,2}\Delta^2 u,$$

the $\Delta^2 u$ being either $\Delta^2 u_4$ or $\Delta^2 u_3$. Similarly in the second of the two cases the u's involved are u_3 and u_4, and the next value of $F(x)$ is

$$u_4+(x-4)\Delta u_3+\frac{(x-3)(x-4)}{1\,.\,2}\Delta^2 u,$$

the $\Delta^2 u$ being either $\Delta^2 u_3$ or $\Delta^2 u_2$. And so on.

13. The ordinary central-difference formulæ are easily constructed by means of the law of formation obtained above.

(i) Suppose that we have an even number of u's, the middle two being u_0 and u_1. Then the central zig-zag formula, starting from u_0, is

$$u_0+x\Delta u_0+\frac{x(x-1)}{1\,.\,2}\Delta^2 u_{-1}$$
$$+\frac{(x+1)x(x-1)}{1\,.\,2\,.\,3}\Delta^3 u_{-1}+\dots$$

The $(2r+1)$th and $(2r+2)$th terms are together

$$\frac{(x+r-1)(x+r-2)\dots(x-r)}{1\,.\,2\dots(2r)}\Delta^{2r}u_{-r}$$
$$+\frac{(x+r)(x+r-1)\dots(x-r)}{1\,.\,2\dots(2r+1)}\Delta^{2r+1}u_{-r}.$$

Replacing $\Delta^{2r}u_{-r}$ by

$$\frac{1}{2}(\Delta^{2r}u_{-r+1}+\Delta^{2r}u_{-r})-\frac{1}{2}(\Delta^{2r}u_{-r+1}-\Delta^{2r}u_{-r})$$
$$=\frac{1}{2}(\Delta^{2r}u_{-r+1}+\Delta^{2r}u_{-r})-\frac{1}{2}\Delta^{2r+1}u_{-r},$$

the two terms become

$$\frac{(x+r-1)(x+r-2)\dots(x-r)}{1\,.\,2\dots(2r)}\cdot\frac{1}{2}(\Delta^{2r}u_{-r+1}+\Delta^{2r}u_{-r})$$
$$+\left(x-\frac{1}{2}\right)\frac{(x+r-1)(x+r-2)\dots(x-r)}{1\,.\,2\dots(2r+1)}\Delta^{2r+1}u_{-r}.$$

(ii) Next suppose that we have an odd number of u's, the middle one being u_0. Then the $(2r)$th and $(2r+1)$th terms of the central zig-zag formula involving u_0 and Δu_0 are together

$$\frac{(x+r-1)(x+r-2)\ldots(x-r+1)}{1.2\ldots(2r-1)}\Delta^{2r-1}u_{-r+1}$$
$$+\frac{(x+r-1)(x+r-2)\ldots(x-r)}{1.2\ldots(2r)}\Delta^{2r}u_{-r}.$$

Replacing $\Delta^{2r-1}u_{-r+1}$ by

$$\frac{1}{2}(\Delta^{2r-1}u_{-r+1}+\Delta^{2r-1}u_{-r})+\frac{1}{2}\Delta^{2r}u_{-r},$$

this becomes

$$\frac{(x+r-1)(x+r-2)\ldots(x-r+1)}{1.2\ldots(2r-1)}\cdot\frac{1}{2}(\Delta^{2r-1}u_{-r+1}$$
$$+\Delta^{2r-1}u_{-r})$$
$$+x\cdot\frac{(x+r-1)(x+r-2)\ldots(x-r+1)}{1.2\ldots(2r)}\Delta^{2r}u_{-r}.$$

14. We have so far (*i.e.* from § 6 onwards) been dealing with the case in which the u's correspond to values of x proceeding by equal increments 1. The method may be adapted to obtain the more general formula in which the u's are $u_a, u_b, u_c, \ldots$, corresponding to $x=a, x=b, x=c, \ldots$; these values of x not necessarily being in order of magnitude.

A rather close examination of the argument in § 11 is required to see where it fails when for 1, 2, 3, ... as the values of x we substitute $a, b, c, \ldots$ It will be found that the critical point is at the statement that "since $F(x)$ is of degree 5 in x, the 6th difference of the sequence $F(a), F(b), F(c), \ldots$ is zero"; for this is not true when the successive differences are defined in the ordinary way. The third differences of the sequence 0, 1, 4, 9, 16, 25, 100, 121, 144, ..., for instance, are not all zero. We might proceed as follows.

15. Suppose that we have 4 u's, viz. u_b, u_c, u_d, u_e, corresponding to $x=b, x=c, x=d, x=e$. We take our interpolation-formula to be

$$u_x=F(x),$$

where $F(x)$ is a polynomial of degree 3 which is equal to u_x for $x=b, c, d, e$. Thus, writing

$$F(x)\equiv p_0+p_1x+p_2x^2+p_3x^3,$$

we have 4 equations

$$\left.\begin{aligned} p_0+p_1b+p_2b^2+p_3b^3 &= u_b \\ p_0+p_1c+p_2c^2+p_3c^3 &= u_c \\ p_0+p_1d+p_2d^2+p_3d^3 &= u_d \\ p_0+p_1e+p_2e^2+p_3e^3 &= u_e \end{aligned}\right\}$$

to determine the p's. These equations will give the p's as linear functions of the u's, so that

$$F(x)=X_bu_b+X_cu_c+X_du_d+X_eu_e,$$

where each X is a polynomial in x of degree not exceeding 3.

16. The conditions determining X_b are that it should be equal to 1 for $x=b$ and be 0 for $x=c, d, e$; so that obviously

$$X_b=\frac{(x-c)(x-d)(x-e)}{(b-c)(b-d)(b-e)},$$

with corresponding expressions for X_c, X_d, X_e. This gives the ordinary Lagrangian formula

$$F(x)=\frac{(x-c)(x-d)(x-e)}{(b-c)(b-d)(b-e)}u_b+\frac{(x-b)(x-d)(x-e)}{(c-b)(c-d)(c-e)}u_c$$
$$+\frac{(x-b)(x-c)(x-e)}{(d-b)(d-c)(d-e)}u_d+\frac{(x-b)(x-c)(x-d)}{(e-b)(e-c)(e-d)}u_e.$$

This is the formula for 4 u's, and it is easily generalized.

17. Now suppose that we take in another u, which may be either u_f or u_a. Let $F(x)$ become $\Phi(x)$, which will be a polynomial of degree 4; and let us write

$$\Phi(x)\equiv F(x)+\phi(x).$$

Then we want to find $\phi(x)$.

First, suppose that the new u is u_f. Then

$$F(b)=u_b,\ F(c)=u_c,\ F(d)=u_d,\ F(e)=u_e,$$
$$\Phi(b)=u_b,\ \Phi(c)=u_c,\ \Phi(d)=u_d,\ \Phi(e)=u_e,\ \Phi(f)=u_f.$$

Hence

$$\phi(b)=0,\ \phi(c)=0,\ \phi(d)=0,\ \phi(e)=0,\ \phi(f)=u_f-F(f).$$

Since $\phi(x)$ is of degree 4 in x, this gives

$$\phi(x)=\frac{(x-b)(x-c)(x-d)(x-e)}{(f-b)(f-c)(f-d)(f-e)}\{u_f-F(f)\}.$$

Let us write this

$$\phi(x) = (x-b)(x-c)(x-d)(x-e)R_f.$$

Then, to find R_f, we observe that, since $F(x)$ is of degree 3 in x, and $\Phi(x)$ of degree 4, the only term in x^4 which occurs in $\Phi(x)$ is in $\phi(x)$. Hence R_f, which is the coefficient of x^4 in $\phi(x)$, is equal to the coefficient of x^4 in $\Phi(x)$. But we have already, in § 16, found the general formula for $F(x)$ or $\Phi(x)$ in terms of the u's; and we see that

$$R_f = \frac{u_b}{(b-c)(b-d)(b-e)(b-f)} + \frac{u_c}{(c-b)(c-d)(c-e)(c-f)} + \dots$$
$$\dots + \frac{u_f}{(f-b)(f-c)(f-d)(f-e)}.$$

Similarly, if the new u is not u_f but u_a, we shall have

$$\phi(x) = (x-b)(x-c)(x-d)(x-e)R_a,$$

where R_a is a corresponding expression involving

$$u_a,\ u_b,\ u_c,\ u_d,\ u_e.$$

18. We might have obtained the value of R_f or R_a, as follows, without using the Lagrangian formula.

We have found that

$$R_f = \frac{u_f - F(f)}{(f-b)(f-c)(f-d)(f-e)}.$$

But $F(x)$, and therefore $F(f)$, is a linear compound of u_b, u_c, u_d, u_e; and therefore R_f is a linear compound of u_b, u_c, u_d, u_e, u_f, in which the coefficient of u_f is

$$1 \div \{(f-b)(f-c)(f-d)(f-e)\}.$$

Now R_f, as stated above, is the coefficient of x^4 in $\Phi(x)$. If

$$\Phi(x) \equiv q_0 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4,$$

then q_4 is obtained from a set of equations

$$q_0 + q_1 b + q_2 b^2 + \dots = u_b,$$
$$q_0 + q_1 c + q_2 c^2 + \dots = u_c, \text{ \&c.}$$

The value of q_4 obtained from these equations is not affected by writing them in a different order, *e.g.*,

$$q_0 + q_1 c + q_2 c^2 + \dots = u_c,$$
$$q_0 + q_1 b + q_2 b^2 + \dots = u_b, \text{ \&c.}$$

Hence q_4 must be a symmetrical linear compound of the u's. We know the coefficient of u_f in q_4; by interchange of letters we get the coefficients of u_b, &c., and thus the general expression for $q_4(=R_f)$ as given in § 17.

19. Thus we may write, if the new u is u_f,

$$\Phi(x)=F(x)+(x-b)(x-c)(x-d)(x-e)\{b,\ c,\ d,\ e,\ f\},$$

where $\{b, c, d, e, f\}$ is a certain linear compound of the u's which is not affected by altering the order of b, c, d, e, f. Or, if the new u is u_a,

$$\begin{aligned}\Phi(x)&=F(x)+(x-b)(x-c)(x-d)(x-e)\{b,\ c,\ d,\ e,\ a\}\\&=F(x)+(x-b)(x-c)(x-d)(x-e)\{a,\ b,\ c,\ d,\ e\}.\end{aligned}$$

This gives the general formula. Starting with u_a, and taking in successively u_b, u_c, . . . , the general value of $F(x)$ is of the form

$$F(x)=u_a+(x-a)\{a,\ b\}+(x-a)(x-b)\{a,\ b,\ c\}+\ldots,$$

where, in each $\{\quad\}$, the order of the letters is immaterial. We now only require a derivation-formula for obtaining $\{a, b\}$, $\{a, b, c\}$, . . . When these have been obtained, the initial order of the u's is immaterial.

20. We can begin with the simplest cases.

(i) For one u, viz. u_a, we have

$$F(x)=u_a.$$

(ii) For u_a and u_b we find that

$$F(x)=u_a+\frac{x-a}{b-a}(u_b-u_a);$$

so that

$$\{a,\ b\}=\frac{u_b-u_a}{b-a}.$$

(iii) Taking in u_c, the additional term will, by § 17, be

$$(x-a)(x-b)\{a,\ b,\ c\},$$

where

$$\{a,\ b,\ c\}=\frac{u_a}{(a-b)(a-c)}+\frac{u_b}{(b-a)(b-c)}+\frac{u_c}{(c-a)(c-b)}.$$

We already know that

$$\{a, b\} = \frac{u_b - u_a}{b-a}, \ \{b, c\} = \frac{u_c - u_b}{c-b};$$

and thence we find that

$$\{a, b, c\} = \frac{\{a, b\} - \{b, c\}}{a-c}.$$

(iv) This leads us to the general law, easily proved by induction, that

$$\{a, b, c, \ldots f, g, h\}$$
$$= \frac{\{a, b, c, \ldots f, g\} - \{b, c, \ldots f, g, h\}}{a-h}.$$

21. Finally, we may return to the point mentioned in § 14, that, when $F(x)$ is a polynomial of degree n in x, the $(n+1)$th difference of the sequence $F(a)$, $F(b)$, ... is not necessarily zero; and we may consider how the differences should be modified in order that the $(n+1)$th should be zero. This gives us an alternative method of obtaining the above results.

(i) First take $F(x) \equiv A + Bx$. Then the terms and their ordinary 1st differences are—

x	$F(x)$	1st diff.
a	$A+Ba$	
		$B(b-a)$
b	$A+Bb$	
		$B(c-b)$
c	$A+Bc$	
		$B(d-c)$
d	$A+Bd$	
$\vdots$	$\vdots$	$\vdots$

Hence, if we divide the 1st differences by $b-a$, $c-b$, $d-c$, ... we get B in each case. Denoting the "divided" difference obtained in this way by $\nabla F(x)$, we get the table—

x	$F(x)$	$\nabla F(x)$
a	$A+Ba$	
		B
b	$A+Bb$	
		B
c	$A+Bc$	
		B
d	$A+Bd$	
$\vdots$	$\vdots$	$\vdots$

(ii) Next take $F(x) \equiv x^2$; the numerical coefficient being omitted. For tabulation at intervals 1, the ordinary 2nd difference would be 2. The 1st divided differences, and their differences, are given by the table—

x	$F(x)$	$\nabla F(x)$	$\Delta\nabla F(x)$
a	a^2		
		$a+b$	
b	b^2		$c-a$
		$b+c$	
c	c^2		$d-b$
		$c+d$	
d	d^2		$\vdots$
$\vdots$	$\vdots$	$\vdots$	

Hence to get a constant 2nd difference we must divide the quantities in the last column by $c-a$, $d-b$, . . . ; and each will then become 1.

(iii) Applying the same method to x^3 and x^4, we get the tables of divided differences—

x	$F(x)$	$\nabla F(x)$	$\nabla^2 F(x)$	$\nabla^3 F(x)$
a	a^3			
		a^2+ab+b^2		
b	b^3		$a+b+c$	
		b^2+bc+c^2		1
c	c^3		$b+c+d$	
		c^2+cd+d^2		1
d	d^3		$c+d+e$	
		d^2+de+e^2		$\vdots$
e	e^3		$\vdots$	
$\vdots$	$\vdots$	$\vdots$		

x	$F(x)$	$\nabla F(x)$	$\nabla^2 F(x)$	$\nabla^3 F(x)$	$\nabla^4 F(x)$
a	a^4				
		$a^3+a^2b+ab^2+b^3$			
b	b^4		$a^2+b^2+c^2+bc+ca+ab$		
		$b^3+b^2c+bc^2+c^3$		$a+b+c+d$	
c	c^4		$b^2+c^2+d^2+cd+db+bc$		1
		$c^3+c^2d+cd^2+d^3$		$b+c+d+e$	
d	d^4		$c^2+d^2+e^2+de+ec+cd$		$\vdots$
		$d^3+d^2e+de^2+e^3$		$\vdots$	
e	e^4		$\vdots$		
$\vdots$	$\vdots$	$\vdots$			

In each of these cases, as in the cases in (i) and (ii), the last divided difference obtained is constant; and the next divided difference, obtained on the same principle, will be zero.

22. The above tables suggest the general proposition that $\nabla^r a^n$, obtained in this way, is the sum of the homogeneous products of degree $n-r$ of the $r+1$ quantities $a, b, c, \ldots$; *i.e.* that

$$\begin{aligned}\nabla^r a^n = a^{n-r} + b^{n-r} + c^{n-r} + \ldots \\ + a^{n-r-1}(b+c+\ldots) + b^{n-r-1}(a+c+\ldots) + \ldots \\ + a^{n-r-2}(bc+\ldots) + b^{n-r-2}(ac+\ldots) + \ldots \\ + \ \&c.\end{aligned}$$

But this is the same thing as saying that

$$\nabla^r a^n = \text{coeff. } \theta^{n-r} \text{ in } \frac{1}{1-a\theta}\cdot\frac{1}{1-b\theta}\cdot\frac{1}{1-c\theta}\cdot\ldots(r+1 \text{ factors}).$$

Let us assume that this is true, the $r+1$ values of x being $a, b, c, \ldots f$, and the next one being g. Then

$$\nabla^r a^n = \text{coeff. } \theta^{n-r} \text{ in } \frac{1}{1-a\theta}\cdot\frac{1}{1-b\theta}\cdot\frac{1}{1-c\theta}\cdots\frac{1}{1-f\theta},$$

$$\nabla^r b^n = \text{coeff. } \theta^{n-r} \text{ in } \frac{1}{1-b\theta}\cdot\frac{1}{1-c\theta}\cdots\frac{1}{1-f\theta}\cdot\frac{1}{1-g\theta}.$$

Hence

$$\begin{aligned}\nabla^{r+1} a^n &= \frac{\nabla^r b^n - \nabla^r a^n}{g-a} \\ &= \text{coeff. } \theta^{n-r} \text{ in } \frac{1}{g-a}\left\{\frac{1}{1-g\theta} - \frac{1}{1-a\theta}\right\}\frac{1}{1-b\theta}\cdot\frac{1}{1-c\theta}\cdots\frac{1}{1-f\theta} \\ &= \text{coeff. } \theta^{n-r-1} \text{ in } \frac{1}{1-a\theta}\cdot\frac{1}{1-b\theta}\cdot\frac{1}{1-c\theta}\cdots\frac{1}{1-f\theta}\cdot\frac{1}{1-g\theta}.\end{aligned}$$

Thus, if the theorem is true for $\nabla^r a^n$, it is true for $\nabla^{r+1} a^n$. But it is true for ∇a^n, which is

$$\frac{b^n - a^n}{b-a} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}.$$

Hence it is true generally.

It follows that

$$\nabla^n a^n = \text{coeff. } \theta^0 \text{ in } \frac{1}{1-a\theta}\cdot\frac{1}{1-b\theta}\cdot \ldots (n+1 \text{ factors})$$
$$=1;$$

and that $\nabla^r a^n = 0$ if $r > n$. Having established these properties, we could obtain the formula

$$F(x) = u_a + (x-a)\nabla u_a + (x-a)(x-b)\nabla^2 u_a + \ldots,$$

or the corresponding zig-zag formula, by the method used in § 11.

National Insurance Acts. The Departmental Committee's Interim Report.

THE "Interim Report of the Departmental Committee on Approved Societies Finance and Administration" which has recently been published (Cd. 8251) is of great interest not only to Actuaries who are interested in the working of National Insurance, but also, owing to the suggestive nature of its proposals, to those members of the profession who are responsible for advising Friendly Societies whose organization is on the general lines of that of the Affiliated Orders.

It will be remembered that the Committee which was set up under the Chairmanship of Sir Gerald Ryan and included amongst its members Mr. Ernest Woods, the President of the Institute of Actuaries, and Sir Alfred Watson, the Chief Actuary to the National Health Insurance Joint Committee, was appointed "to consider and report upon any amendments in the financial scheme of the National Insurance Acts which experience of the administration of sickness, disablement and maternity benefits may suggest as desirable, within the existing limits of contributions and benefits and apart from further Exchequer Grants, before the completion of any valuation of Approved Societies; and, further, to consider how far the work of Approved Societies could be simplified and its cost reduced, without detriment to the interests of insured persons, by amendment of the Acts and Regulations; and to make recommendations thereon." The Report embodies their recommendations in regard to the first part of the reference.

Before dealing with the terms of the Report it may be convenient to recapitulate the various stages whereby the present actuarial basis of the Acts was reached.

It will be remembered that, under the provisions of the original Bill, Sickness Benefit was payable for the first thirteen weeks (commencing on the fourth day of sickness) ordinarily at the rate of 10*s*. per week for men and 7*s*. 6*d*. per week for women, the benefit being thereafter reduced to 5*s*. per week.

The report of the Actuaries thereon* disclosed a margin in the weekly contributions of ·71*d*. in the case of men and ·67*d*. in the case of women, the equivalent loadings on the rates of sickness and disablement being 24·2 per-cent and 28·0 per-cent respectively. This was exclusive of the value of the relief in respect of the first three days, which it was recommended should be kept as a general margin.

In the Committee stage the full rate of benefit was continued until the end of 26 weeks, the rate of sickness benefit in the case of unmarried minors was increased and certain other less important changes were introduced. The total effect of these changes was to reduce the loadings referred to in the preceding paragraph to 13·3 per-cent and 20·8 per-cent respectively.

It is to be noted that the Actuaries considered these margins as no more than sufficient to guard against the possibility of fluctuations in the sickness rates which might arise not only as a result of temporary causes but also on account of the undoubted tendency of certain classes of hazardous risks to segregate themselves in particular societies.

This point is brought out in paragraph 7 of the report† from which the following is an extract:

"We consider that any saving that may be effected by the sickness benefit not commencing until the fourth day of sickness must be kept as a margin, and without this margin we do not consider that the rates of sickness employed in our calculations are applicable to the conditions of a national scheme as set out in the Bill . . . . we are of opinion . . . . that the whole margin between the estimated contributions and those actually payable under the Bill should be regarded as available to meet the heavier rates of sickness and disablements which must be expected in a certain number of societies. Unless this margin is retained it is probable that a considerable number of societies will show deficiencies on valuation, and this might endanger the success of the scheme."

* Cd. 5681. † Cd. 5983.

The whole position, as affected by the amendments to the scheme which were introduced in the passage of the Bill through the House of Commons, was again reviewed by the Actuarial Advisory Committee which was set up under the Acts, when the basis of the calculation of Reserve Values came to be considered.

In their Reports* it was indicated that, after making appropriate allowance for the value of the first three days of sickness, a margin in the contributions remained which the Committee recommended should be applied as an equal loading of 12·745 per-cent to the rates of sickness and disablement in the case of men, the basis of the calculations being the Manchester Unity Whole Society rates. In the case of women the corresponding loadings were 12·745 per-cent to the sickness rates and 19·17 per-cent to the disablement rates. Bearing in mind that the Manchester Unity Sickness rates contain provision for the payment of benefit in cases of incapacity due to accident and industrial disease, the actual loading over the Manchester Unity standard of sickness was about 24 per-cent.

Stress was again laid on the importance of maintaining the margins to provide against the possible effect on the sickness rates of segregation (which was clearly revealed in the actual grouping of insured persons among the various Approved Societies) and of the bringing into insurance for the first time of large numbers of people who were unaccustomed to this form of mutual thrift.

Turning now to the actual experience of Approved Societies as set out in the Report of the Ryan Committee, it will be seen from the following Table that in the years 1913 and 1914 in the case of men the rates of sickness experienced were such as to

	Men			Women		
	1913	1914	1915	1913	1914	1915
	d.	*d.*	*d.*	*d.*	*d.*	*d.*
Actual expenditure on sickness and maternity benefits (in pence per week per member)	2·76	3·03	2·76	2·51	2·60	2·04
Corresponding expected expenditure	2·80	3·00	3·00	1·78	1·85	1·85

* Report for 1912–13 on the Administration of the National Insurance Act, Part 1 (Cd. 6907), pp. 552–60.

result in the immediate absorption of the above margin, while in the case of women the rates of sickness for which provision had been made were materially exceeded.

The experience of the year 1915 is included as a matter of interest, but it can hardly be considered as a fair basis of comparison owing to the abnormal industrial conditions arising out of the War.

The problem with which the Committee had to deal was, therefore, virtually to restore to the sickness rates such margins over the average experience as might reasonably be considered adequate to ensure the successful working of the scheme, and at the same time to deal effectually with the special excess of sickness which had been revealed in the case of married women.

In connection with the general excess of sickness among women it is important to consider whether this excess is of such a character as to affect the women's reserve values; in other words, whether the excess is in the nature of a constant addition to the Sickness rates at each age, or whether it varies from age to age.

The Committee are informed that on an investigation into the experience of a considerable number of women drawn from Societies of various types, clear indications are found that the excess of sickness among women does not increase with age. Their recommendations accordingly do not involve an alteration in reserve values. This means, of course, that the whole of any new money which can be found for societies may be regarded as a provision for current excess instead of part of it (*i.e.*, the net premium for the excess at age 16) being so applied and part of it used to create the additional reserves which an excess increasing with age would require. But having regard to the heavy lapse of women from insurance on marriage the point would not appear in any case to be of more than theoretical interest.

In considering the remedies available under the present law the Committee rightly emphasize the importance of the maintenance by Societies of an efficient system of administration.

Apart from this, the Acts provide in the case of deficiencies revealed on valuation certain measures for the redemption of such deficiencies by the pooling of funds (within certain limits) in the case of small societies and branches. Section 63 of the Act, which provides for enquiries into causes of excessive sickness where insanitary conditions are alleged against an employer or a local authority, has also an obvious bearing on valuation deficiencies.

In addition there remain the general provisions of Section 38 of the Act of 1911 with reference to the redemption of these deficiencies by the reduction of benefits or the imposition of levies.

The Committee reject this as a sufficient solution of the question for the following reasons:

"We interpret our reference to mean that the task entrusted to us is to suggest remedies which will so far as possible obviate the necessity of resort to these provisions. But in any event we could not hold that it would be equitable to allow the Act to take its course in all cases of deficiency without qualification. We may refer, for instance, to the position, in this connexion, of women's Societies and branches. One of the most important effects of the operation of the Acts has been to increase the knowledge previously available as to the health of women engaged in insurable employment; and it is possible that had Parliament been in possession of such knowledge in 1911, the conditions as to women's insurance would have been materially different. Further, on practical grounds we could not recommend that persons who were invited to associate voluntarily in particular Societies should be required to bear the consequences of an act of choice necessarily performed on incomplete information, without the possibility of having these consequences mitigated."

They also state that they consider themselves to be precluded by their terms of reference from seeking a remedy in the direction of new grants from the Exchequer, of an increase in the present rates of contribution, or of a reduction in the standard rates of benefit. They are, moreover, unable to entertain the suggestion that the whole or a proportion of the surpluses accumulated by certain Societies should be applied to the redemption of deficiencies arising in Societies less fortunately situated.

They are accordingly led to the conclusion that a remedy should be sought by applying to this purpose a certain portion of the moneys which are at present allocated to the Sinking Fund for the redemption of Reserve Values.

The Committee are advised that the proportion of the women's contributions applied to this purpose is excessive, and that to correspond with the reduction of $1\frac{5}{8}d.$ made in the case of men, the deduction in the case of women should be about $1\frac{1}{4}d.$ instead of $1\frac{1}{2}d.$

Accordingly the proportion of the contribution proposed to be released from the Sinking Fund is in the case of women $\frac{3}{4}d.$ per contribution and in the case of men $\frac{5}{8}d.$ This would leave for interest on, and redemption of, reserve values, $\frac{3}{4}d.$ in the former case and $1d.$ in the latter and would have the effect of

bringing the men's and the women's contributions to the Sinking Fund into proper relationship.

The effect of these proposals would be to reduce the present income of the Sinking Fund from £4,500,000 to £2,700,000 a year and, *ipso facto*, to extend the period of redemption of reserve values, with the result that the extension of benefits, which is contemplated when reserve values have been redeemed, would be deferred.

The Committee consider that this disadvantage is more than met by the following facts:

"(*a*) The scheme which we propose to recommend will leave unimpaired the rights of members of Societies with a disposable surplus to the immediate enjoyment of the whole of that surplus, and will enlarge the rights in this respect of branches, (including Societies of men and women insured through separate Benefit Funds), and small Societies which are subject to pooling liabilities.

(*b*) The scheme will greatly diminish even if it may not in some cases wholly remove, the prospect that the members of particular Societies will have their benefits reduced as the result of a deficiency emerging upon valuation.

(*c*) The scheme offers a substantial prospect to many Societies of the provision of increased additional benefits at a date considerably in advance of 1932."

In considering the application of the moneys so released the Committee state that their object has been to frame such a scheme as will enable each Society, so far as possible, to meet its own liabilities, and they conclude that such a scheme should contain five principal features:

"(*a*) It should provide an immediate increase in the income available to Societies for the payment of benefits to women generally.

(*b*) It should make further special provision for the claims falling upon Societies in respect of married women.

(*c*) It should provide, as a measure of precaution, for the accumulation of a Special Reserve Fund to meet if necessary, the indirect but possibly prolonged effects of war service upon the general sickness rate of the male insured population.

(*d*) It should place in the hands of all Societies, whether composed of men or women, or partly of one sex and partly of the other, additional moneys to be available to meet contingent liabilities which may arise in any Society owing to excess of claims for sickness, disablement, or maternity benefits. These moneys should be available for immediate application in cases of deficiency on valuation.

(*e*) It should include provision for further deficiency due to the special risks to which the funds of certain Societies are exposed owing to the inclusion in their membership of an abnormal number of persons who are below the average in health, as the result either of hazardous occupations, unhealthy environment, or an exceptionally low standard of living."

In giving effect to the above considerations it is recommended in the case of women that $\frac{1}{3}d.$ out of the released $\frac{3}{4}d.$ per contribution (or £280,000 in all) should be applied to increasing the current income of the Benefit Funds, in part directly, and in part through the medium of the Women's Equalization Fund.

The amount available for the former purpose is estimated to be £145,000, the proportion to be carried to the Women's Equalization Fund (namely £135,000) being estimated as follows:

It is assumed in the first place that to this £135,000 will be added an equal amount from moneys provided by Parliament, this sum representing the equivalent of the grant in aid at present made in recognition of the cost to Societies of paying benefits arising out of pregnancy.

It should be remembered that as regards the cost of benefits, two-ninths in the case of men and one-fourth in the case of women is defrayed by moneys provided by Parliament. Any cash sums therefore which are made available towards the payment of benefits will attract a State Grant to the extent of two-sevenths (or one-third) of their values and the total provision when expressed in the form of benefit payments is increased accordingly.

It follows that the sum of £135,000 to be carried to the Women's Equalization Fund out of contributions will attract an Exchequer Grant of one-third, making it £180,000; similarly the like amount which it is assumed will be paid into that Fund from the Exchequer will itself attract a further Exchequer Grant of one-third, again making £180,000.

The Committee recommend that the Fund should be a single fund, established for the whole of the United Kingdom, inasmuch as the proportion of Married Women varies in different localities, and moreover a large number of societies, which operate in more than one part of the United Kingdom, insure their women members in a common Benefit Fund.

With regard to the disposal of the Fund the Committee are informed "that, on the best information available and after making due allowance for the difference in age, the claims of married women for sickness benefit are in excess of those of unmarried women to an average extent not exceeding one week per woman per annum."

This excess includes a certain amount due to claims arising out of pregnancy, but although this is not a liability specifically provided for in the rates of sickness employed in the calculation

of Reserve Values, the Committee are unable to recommend that such amounts should be directly recouped to Societies, as the resulting complexities in accounting and tabulation would be considerable, and difficult questions as to the supervision of the claims for benefit by Societies (which would have ceased to be interested in the amount expended) would arise.

Apart from the excessive sickness there are indications of excess over the expectation in the case of maternity claims of married women. It is pointed out that this does not necessarily mean that their fertility has been underestimated, the probability being that a difficulty is experienced by Societies in determining the point of time at which a married woman who has ceased work has definitely left employment, and has consequently become subject to suspension from the ordinary benefits.

A flat rate of distribution in proportion to the number of married women in each Society is rejected on three grounds: first, that it would not give effect to variations in the age distribution; secondly, that it would fail to meet the varying incidence of pregnancy sickness due to differences between the birth rates of different sections of the community: and thirdly, that it would not secure to individual societies a proper recoupment in respect of the claims of pregnant unmarried women.

The Committee think "that a practical solution of the difficulty can be obtained by dividing the Fund into two parts, to be distributed respectively on the basis of the number of married women members in each Society, and on the basis of the cost of maternity benefit. Under the first of these heads Societies could be supplied with a sufficient additional income in respect of married women of all ages to meet the excess of claims which appears to arise independently of the incidents of maternity. Under the second head Societies could be supplied with sufficient additional funds to provide suitably for the extra cost of pregnancy sickness cases and for any addition to the normal cost of maternity benefit resulting from the cause named above. Further, by suitably distributing the income of the Fund under these two heads, an automatic provision could be made for cases of incapacity due to early miscarriage or abortion, which are not readily traceable in the accounts of Societies."

They accordingly recommend that the grants from the Women's Equalization Fund should be made in two parts, consisting of:

(*a*) A grant in respect of each employed married woman

of 3*s*. 6*d*. a year payable to Societies in proportion to the number of married women included in their membership (approximate present cost £110.000).

(*b*) A grant equivalent to three-quarters of the amount paid by Societies in maternity benefit in respect of women's insurance (estimated present cost £160,000).

As these Grants are to be made to the Benefit Funds of societies it follows that they will be applied in defraying the *net* cost of benefits, *i.e.*, after deduction of the Exchequer Grant of one-fourth.

In the case of men it has been necessary to give consideration to the possible after-effects of the War on the funds of Approved Societies. On the one hand there is the possibility that the arduous labours now being borne by a large proportion of the industrial community will react unfavourably upon the future sickness rates, on the other hand, as the Committee point out, beneficial effects should flow from the general extension of physical training. There are in addition certain special sources of profit, such as the exceptional release of reserves by death, and a general rise in the rate of interest consequent upon the War which must be set against the additional liabilities in which societies may become involved.

The Committee are satisfied, however, that prudence dictates the formation of a Special Reserve Fund, to accumulate for a period of 10 years so as to be available to meet hereafter any increase of claims which may emerge as a result of War service. To this Fund they recommend that two-ninths of the $\frac{5}{9}d$. diverted from the men's contribution to the Sinking Fund, should be carried, this yielding at the present time about £250,000 a year.

After allowing for the above special factors there remains out of the moneys diverted from the Sinking Fund an amount of $\frac{5}{12}d$. per contribution in the case of women and $\frac{35}{81}d$. per contribution in the case of men. These sums which become equal in amount, namely $\frac{5}{9}d$. when increased by the State Grant of one-third for women and two-sevenths for men, have to be allotted between the "Contingencies" and the "Special Risks" Funds respectively.

In considering this allocation it is important to recognize that the Contingencies Fund is designed to provide a measure of reinsurance within the Society itself, whereas the Special Risks Fund operates as a reinsurance Fund for all societies throughout the United Kingdom. Societies will, therefore, have a direct

interest in their own Contingencies Fund, inasmuch as any unexpended balance would be available as surplus for the provision of additional benefits.

In order, therefore, to maintain and develop the interest of insured persons in the management of their Societies the Committee recommend that the Contingencies Funds should be built up from a proportion of not less than three-quarters of the amount remaining available for the joint purposes of the Contingencies and Special Risks Funds.

Assuming for the purposes of illustration that the Contingencies Funds will be credited with four-fifths of the total amount available, the annual income so provided would be at the present time about £710,000 in respect of men and about £280,000 in respect of women.

In the following tables summarizing the foregoing proposals, the first shows the proposed allocation of the moneys derived from Sinking Fund sources and the second shows the general financial effect of the proposals by a comparison of the provision made for benefits under the Scheme (expressed in the form of pence per member per week) with the actual expenditure in various years.

—	Men		Women	
	At the Rate of Pence per Contribution	Total Annual Amount	At the Rate of Pence per Contribution	Total Annual Amount
	d.	£	*d.*	£
1. To Benefit Fund	...	...	·17	145,000
2. „ Women's Equalization Fund	...	...	·16	135,000*
3. „ Men's Special Reserve Fund	·12	250,000	...	...
4. „ Contingencies Fund ...	·35	710,000	·34	280,000
5. „ Special Risks Fund... ...	·09	180,000	·08	70,000
Total	$\frac{5}{9}$ = ·56	1,140,000	$\frac{3}{4}$ = ·75	630,000
The appropriation of the above items would leave for interest and redemption of reserve values—				
6. Interest	1·00	1,500,000	·75	460,000
7. Redemption of Reserve Values		560,000		170,000
Grand total	$1\frac{5}{9}$ = 1·56	3,200,000	$1\frac{1}{2}$ = 1·50	1,260,000

* It is assumed that Parliament will be asked to provide a grant of an equivalent amount.

—	Men			Women		
	1913	1914	1915	1913	1914	1915
	d.	*d.*	*d.*	*d.*	*d.*	*d.*
Present position (under the Acts of 1911 and 1913)	2·80	3·00	3·00	1·78	1·85	1·85
Additional income to be credited direct to Benefit Fund	...	...	...	·20	·20	·20
Addition from Women's Equalization Fund	...	...	...	·37	·37	·37
Total, being normal provision under the Scheme	2·80	3·00	3·00	2·35	2·42	2·42
Addition from Contingencies Fund, available for immediate use in deficiency cases	·38	·38	·38	·38	·38	·38
	3·18	3·38	3·38	2·73	2·80	2·80
Average actual cost	2·76	3·03	2·76	2·51	2·60	2·04

As an explanation of the manner in which the figures in the latter table are obtained, the following calculation of the figure of ·38*d.* representing the value of the contribution from the Men's Contingencies Fund, may be taken.

As explained above, the total amount available from each contributor, for the purposes of the Contingencies and Special Risks Funds is $\frac{35}{81}d$. Taking four-fifths of this as applicable to the former Fund, a sum of $\frac{28}{81}d.$, or ·346*d.* is obtained. Owing to the fact that contributions are not payable during sickness and that accordingly loss of contributions as well as excess sickness has to be met, the amount actually available for benefits is reduced to ·327*d.* On the basis of a payment of 48 contributions a year on the average (four weeks being omitted on account of sickness and unemployment) the annual sum so produced is 15·7*d.* which, when increased by the State Grant of two-sevenths becomes 20·2*d.*; and this in an average year consisting of $52\frac{1}{5}$ weeks reduces to the weekly amount of ·38*d.* as stated.

With reference to the accumulation and disposal of the Contingencies Fund, the Committee lay down the following general principles, subject to certain modifications in the case of societies of particular types:

"(*a*) A Contingencies Fund should be set up in respect of each

Society, and should be applied solely for the benefit of members of that Society.

(*b*) On the completion of a valuation of a Society, if a deficiency should appear, a sum equal to the deficiency or to the balance of the Contingencies Fund (whichever were the smaller) should be transferred from the Contingencies Fund to the Benefit Fund; any balance remaining in the Contingencies Fund after this transfer had been made should be carried forward and be recruited year by year by contributions as above explained, and by interest at the average rate earned.

(*c*) Should the balance of the Fund, after the completion of a valuation (other than the first valuation), and the discharge of any liability to meet a deficiency as indicated in (*b*), exceed a limit to be prescribed, which it is suggested should be five times the sum paid into the Fund out of contributions in the calendar year preceding the valuation date, the excess should be paid into the Benefit Fund of the Society, and become available as an addition to the surplus for the purpose of Section 37 of the Act of 1911."

The various types of Societies which require consideration are as follows:

(*a*) Societies without branches consisting of men only or women only and not subject to the pooling provisions of the Act of 1911.

(*b*) Societies without branches, consisting of men and women, with a Common Benefit Fund.

(*c*) Societies without branches, consisting of men and women, with separate Benefit Funds.

(*d*) Societies with Branches.

(*e*) Societies without Branches subject to pooling provisions under the Act of 1911.

In the case of Societies coming under the first category, no complexities present themselves and it is recommended that the Scheme be applied without modification.

In the second class of Societies the position is more complicated, but after reviewing the various arguments (which are fully discussed in paragraphs 51 to 59 of the Report) the Committee recommend that a single Contingencies Fund be established to be available to meet any deficiency which the valuation of the Society may disclose.

In the third class of Societies the Committee recommend that two Contingencies Funds be set up and that if the Benefit Fund of one section is in deficiency it be required to resort in the first instance to its own Contingencies Fund; and that if after exhausting that Fund it remains in deficiency, it should

resort to any balance in the Contingencies Fund of the other section of the members, to an extent not exceeding the balance of the contributions paid into the latter Fund since the last valuation.

It should be noted that the general effect of the proposals is to preserve to either section of the membership the whole of any balance of its own Contingencies Fund which may have been carried forward at the previous valuation and also the whole of any surplus which may be found to exist in its own Benefit Fund.

An analogous arrangement is proposed in the case of Societies with branches, the Committee's recommendations being as follows:

"(i) A central Contingencies Fund should be established for the Society, and the appropriate part of the contributions of each member of the Society should be credited to this central Fund.

(ii) Each branch found on valuation to be in deficiency should have its deficiency made good from the Society's Contingencies Fund, subject to the right of the central authority to refuse, with the consent of the Insurance Commissioners, to make good the whole or any part of the deficiency of a branch which has been maladministered.

(iii) The whole of the surplus of a branch which is disclosed on its valuation should be reserved for the use of the branch, the existing provision under which one-third of the disposable surplus is transferable to the central authority for the purpose of meeting deficiencies being repealed.

(iv) Any balance remaining in the Contingencies Fund of the Society after making good deficiencies should be transferred to the branches in proportion to the amounts which their members have respectively contributed to the Fund, reduced by any amounts which may have been paid to particular branches in redemption of deficiency.

(v) The sum paid to a branch under (iv) should be dealt with as follows:—

(*a*) Where it is paid to a branch which in consequence of maladministration has not been relieved out of the Fund to the full extent of its deficiency, the whole sum, or so much of it as is required to make good the remaining deficiency, should be paid into the Benefit Fund of the branch.

(*b*) Subject to (*a*) the amount paid to each branch should be paid into a Contingencies Account in the branch.

(*c*) The sum at the credit of the branch Contingencies Account should accumulate (with interest), and if at any valuation the balance of the Account exceeds five times the amount which the members of the branch have contributed to the Society's Contingencies Fund in the calendar year preceding

the valuation date, the excess should be transferred to the Benefit Fund of the branch and become available for additional benefits.

(*d*) Where, however, a branch which has a sum to the credit of its Contingencies Account (accumulated from distributions following previous valuations) is in deficiency on any valuation, it should be required to apply the balance of its Contingencies Account to making good the deficiency before resorting to the central Contingencies Fund of the Society."

In the case of the societies without branches subject to the pooling provisions of the Act of 1911, the Committee after examining the present position have found it necessary to recommend considerable alterations in the existing scheme.

Section 39 of the Act of 1911 provides that all Societies which at the date of any valuation have less than 5,000 members shall for valuation purposes either—

(*a*) be grouped with other Societies in any association containing in all not less than 5,000 insured persons, which may be formed under the Section ; or

(*b*) in default of joining such an association, be grouped according to the county or county borough in which they carry on business.

One-third of the surplus of any grouped Society is transferred to a central pool in respect of its own group and a grouped Society in deficiency is entitled under certain conditions to have three-quarters (or the whole if the Central Body so determine) of its deficiency made good within the limits of the fund so created.

For the successful working of any scheme of reinsurance of this kind, it is essential that two fundamental conditions should be fulfilled : first, that the scheme should, in fact, reinsure the inequalities which it is intended to correct, and secondly, that the premiums paid should be commensurate with the risk run.

In examining the above scheme it is seen at once that the first condition is not satisfied, for, assuming that in any group of societies the exact average of sickness is realized over the group as a whole, it is obvious that the fund created by the pooling of one-third of the surpluses will be insufficient to redeem more than one-third of the deficiencies. It is understood also that in many cases the pooling area is very limited, descending occasionally to single societies with a very small membership. Apart from this there will be a tendency for strong societies to associate together voluntarily and exclude the weak who will thereby be left to their own unaided resources.

It must be considered, therefore, that the scheme of the 1911 Act fails to provide adequately against the risks of fluctuations in the experience of small societies.

As regards the second condition, the premium paid under the scheme of the 1911 Act, depending as it does upon the amount of disposable surplus, is purely empirical and bears no obvious relation to the risk against which it is intended to provide. Further, if the individual risks were equal and therefore variations in the experience of Societies were due to fluctuations only, theoretically the reinsurance premium paid per member should decrease as the membership of the society increases. As against this there is the admitted difficulty of securing adequate supervision in the case of large societies, but the difficulties in this respect must eventually tend to become uniform when a certain membership is attained.

Even after making allowance for these considerations, there seems no doubt that the risk of divergence from the average experience is relatively smaller in the case of a society of normal constitution with a membership of 5,000 than in the case of a society with a membership of 50. Under the existing scheme of the Acts these societies are both placed on the same footing for pooling purposes. In cases where the larger societies are likely to be in deficiency as a result of occupational or other special risks, the present scheme would tend to place an undue burden on the small societies round them.

The Committee accordingly recommend that all societies with between 1,000 and 5,000 members be exempted from pooling. In arriving at this conclusion the Committee take into account the facts that the Contingencies Funds of Societies will provide a considerable margin for fluctuations and that they recommend at a later stage in the Report the extension of the valuation period from three to five years. This extension would increase the value of the Contingencies Funds as a means of dealing with adverse fluctuations.

The Committee understand that the total number of societies in England at present liable to pool is about 1,300 and that these would be divided under existing conditions into 128 county and county borough groups and an indefinite number of voluntary associations. There is no prospect that with such minute division the pooling arrangements would substantially achieve their purpose, and they recommend therefore that territorial-grouping and unrestricted voluntary association for pooling

purposes be abolished and one national pooling fund formed for each of the four parts of the United Kingdom. The Committee however propose that, subject to certain safeguards, the right of association should be conceded to societies having a common origin or possessing some special community of interest.

As regards the financial arrangements it is recommended that the existing provisions of the Act of 1911 as to the levy on disposable surpluses to meet deficiencies be rescinded and that any society whose membership does not exceed 1,000 which is in deficiency shall in the first instance apply the whole of its Contingencies Fund to meet the deficiency, and that if solvency is not then restored, the society shall be entitled to apply for assistance, up to the full amount of the remaining deficiency, from a fund formed by a levy on the limited Contingencies Funds of other societies of the same class.

In considering the extent of such levy, the Committee draw a distinction between the case of a society with branches and the case of the pooled small societies, which having no common bond are not in a position to exercise control over one another. They, therefore, consider that the call on the limited Contingencies Fund of a society with a membership not exceeding 1,000 should be confined to a maximum of one-half the amount of that Fund.

The above proposals complete the measures which the Committee recommend for adoption in order that societies of a normal type may be enabled to provide against periodical fluctuations in the sickness experience.

There remains the problem of the segregation in individual societies of particular sections of the insured population who by reason of special conditions of physique, occupation or environment, are subject to an abnormal rate of sickness against the risks of which the Special Risks Fund is designed to afford a provision.

It appears that these special risks are not uniformly distributed over the United Kingdom—Wales in particular being heavily burdened in respect of occupational liabilities—and it is accordingly recommended that one fund be established for the whole of the United Kingdom. The grants from this fund would be made only after an investigation into the administration of the applicant society, with the object of securing that grants are only made to the extent to which the deficiency can be shown to be due to abnormal risks and not to lax management.

Having regard to the many uncertainties which may affect the future finance of National Insurance and on which further

light will be thrown by wider experience in the working of the Acts, the Committee recommend that legislative provisions embodying their proposals as to the Contingencies Funds, the Special Risks Fund and the Women's Equalization Fund, should be operative in the first instance for a period of ten years only as from January, 1913, and that at the end of that time the question should be further considered in the light of the experience then available.

The extension of the period of redemption of reserve values involved in the adoption of the above proposals until this review of the position takes place, would be about six years, that is from 1932 to 1938. If the scheme were applied without the foregoing limitations, the resulting extension of the Sinking Fund period would have been about twenty years, that is, to about 1952.

The above virtually complete the Committee's proposals so far as the financial basis of the Acts is concerned. The remainder of the Report is devoted to the discussion of various administrative points of which perhaps the most important are the question of the valuation period (which it is recommended should be extended to 5 years) and the rate of interest to be assumed in the calculations.

In considering whether this rate should be raised to $3\frac{1}{2}$ per-cent it is pointed out that for some years a large proportion of societies' credits will consist of Reserve Values, the interest on which has to be provided from the Sinking Fund moneys: any increase in the interest income in respect of Reserve Values would thus be secured at the expense of the more direct additional provision which the Committee propose should be made. Moreover, even though a higher rate than 3 per-cent may be earned for some time to come, is is not desirable that every margin should be brought into account for present purposes; the Committee point out the many contingencies, *e.g.*, a possible change in the future rate of mortality, which might disturb the finances of approved societies in the absence of any compensating factor such as a limited interest profit would provide.

The Committee, therefore, recommend that no change be made in the rate of interest assumed in the calculations.

It should be noted that special emphasis is laid on the necessity for an actuarial investigation, at the earliest possible date, of the experience of Societies in respect of sickness, disablement and maternity benefits, not only with regard to sex and age, but also, as far as possible, with regard to industrial

conditions. Such an enquiry would, of course. include an examination into the many special factors affecting the insurance of women.

It is not our intention to offer any criticisms of the Report. but it may be permissible to suggest that, within the limits of their terms of reference. the adoption of some such scheme as that put forward by the Committee was almost inevitable. Given the factor of segregation of risks in individual societies, combined with the operation of the flat rate of contribution (concerning the justice of which opinions may differ). the necessity of avoiding widespread valuation deficiencies. and the fact that the original margins in the contributions. which were designed to provide safeguards against these difficulties, were found from the commencement to have been absorbed in payment of sickness claims. the only remaining source of income was that formed by the Sinking Fund moneys and the only practical scheme centred round a broad application of the principles of reinsurance of risks.

The details of such a scheme afford boundless possibilities for the exercise of individual imagination and individual judgment, but the Committee may, at any rate. be congratulated on the adoption of a scheme which will commend itself to Actuaries not only as being eminently practical. but also as affording substantial justice in regard to the problems with which they had to deal.

APPENDIX.

Summary of the Expenditure of Approved Societies on Sickness, Maternity and Disablement Benefits in the Years 1913, 1914 *and* 1915.

Country	No. of Members	Sickness Benefit	Maternity Benefit	Disablement Benefit	All Benefits	Average Expenditure shown in Pence per Week: Sickness Benefit	Maternity Benefit	Disablement Benefit	All Benefits
1913. *Men.*									
		£	£	£	£	*d.*	*d.*	*d.*	*d.*
England ...	7,359,866	3,413,052	938,837	...	4,351,889	2·14	·59	...	2·73
Scotland ...	1,026,649	422,564	137,332	...	559,896	1·90	·62	...	2·52
Ireland ...	457,227	230,119	68,339	...	298,458	2·32	·69	...	3·01
Wales ...	551,219	322,377	78,139	...	400,516	2·70	·65	...	3·35
United Kingdom	9,394,961	4,388,112	1,222,647	...	5,610,759	2·16	·60	...	2·76
1913. *Women.*									
England ...	3,290,807	1,756,610	31,995	...	1,788,605	2·46	·04	...	2·50
Scotland ...	431,404	199,068	7,031	...	206,099	2·13	·08	...	2·21
Ireland ...	221,432	148,145	2,304	...	150,449	3·08	·05	...	3·13
Wales ...	109,465	62,752	1,579	..	64,331	2·63	·07	...	2 70
United Kingdom	4,053,108	2,166,575	42,909	...	2.209,484	2·46	·05	...	2·51
1914. *Men.*									
England ...	7,537,794	3,944,065	956,917	115,958	5,016,940	2·41	·58	·07	3·06
Scotland ...	1,033,276	507,233	144,309	17,443	668,985	2·26	·64	·08	2·98
Ireland ...	474,544	253,809	73,139	3,367	330,315	2·46	·71	·03	3·20
Wales ...	579,948	378,629	84,974	9,650	473,253	3·38	·76	·09	4 23
United Kingdom	9,625,562	5,083,736	1,259,339	146,418	6,489,493	2·43	·60	·07	3·10
1914. *Women.*									
England ...	3,336,285	1,728,104	177,003	36,222	1,941,329	2·39	·24	·05	2·68
Scotland ...	446,337	210,744	17,788	6,775	235,307	2·17	·18	·07	2·42
Ireland ...	231,291	120,341	13,248	1,177	134,766	2 39	·26	·02	2·67
Wales ...	117,912	67,547	2,874	1,588	72,009	2·63	·11	·06	2·80
United Kingdom	4,131.825	2,126,736	210.913	45.762	2,383,411	2·37	·23	·05	2·65
1915. *Men.*									
England ...	*See Note 2.	3,631,223	864,609	457,384	4,953,216	2·21	·53	·28	3·02
Scotland ...		478,073	130,318	69,918	678,309	2·13	·58	·31	3·02
Ireland ...		220,193	65,717	21,672	307,582	2·13	·64	·21	2·98
Wales ...		318,280	75,751	40,948	434,979	2·84	·68	·37	3·89
United Kingdom		4,647,769	1,136,395	589,922	6,374,086	2·22	·54	·28	3·04
1915. *Women.*									
England ...	*See Note 2.	1,335,994	154,318	198,464	1,688,776	1·84	·21	·27	2·32
Scotland ...		170,755	14,719	34,706	220,180	1·76	·15	·36	2·27
Ireland ..		95,765	10,614	10,214	116,593	1·91	·21	·20	2·32
Wales ...		53,659	2,852	8,343	64,854	2·09	·11	·33	2·53
United Kingdom		1,656,173	182,503	251,727	2,090,403	1·84	·20	·28	2·32

* For Notes, see p. 118.

NOTES.

1. In compiling the above statement the figures shown by the returns of societies for the year 1914 have been increased to represent a full year's experience, that benefit year having consisted of 51 weeks in most cases, but of 50 weeks in some.

2. In respect of 1913 the number of members is taken as indicated by the number of contribution cards surrendered for the third and fourth quarters. For 1914 the number of cards surrendered for the first half of 1914 is similarly taken, this supplying an approximation to the numbers "exposed to risk" which is sufficiently accurate for the limited purpose here in view. For 1915 the membership figures are taken as being the same as in 1914. This expedient probably under-estimates the number of lives at risk to a very slight extent, and correspondingly over-estimates the average expenditure. Exact figures of the number of persons insured in 1915 are not yet available.

3. The returns of expenditure are practically complete. As the averages are based on the total membership arrived at as indicated above, appropriate additions have been made to the expenditure figures to provide for the few cases in which returns have not been supplied.

4. Certain adjustments have been necessitated by the withdrawals from ordinary insurance of the men who have joined the naval and military forces. In these cases the right to sickness and disablement benefits has been suspended for the time being, but in lieu thereof an allowance of equivalent value has been included in the reduction of contributions by 4*d*. a week which is made in the case of men serving with the forces. Due attention has been paid to this point in summarizing the expenditure and calculating the weekly average.

5. The payments have been similarly adjusted to give effect to the sums allowed as reduction of contributions in lieu of sickness benefit to insured persons subject to the conditions of Sections 47, 48 and 53.

6. It should be carefully noted, in connection with any comparisons which may be made, on the basis of these tables, between the experiences of different parts of the United Kingdom that, as regards women, a substantial difference arises from the varying proportions in which married women (whose claims are relatively heavy) enter into the total. Thus in England the married women are estimated to be 16 per-cent of the total number of insured women, and in Ireland about 10 per-cent; in Scotland the married women are under 7 per-cent of the total and in Wales are 6 per-cent. From a superficial examination of the tables it might be concluded, for example, that the cost of women's sickness benefit in Scotland in the year 1915 was generally about 5 per-cent below the corresponding cost in England; but no such difference appears when the claims of married and unmarried women are separated. In respect of both men and women the possibility that the claims may be affected to some extent by differences of age distribution should also be kept in view in comparing sectional experiences.

LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law.*

Validity of an assignment of a life policy.

THE validity of an assignment of a policy of assurance, the assignment being conditional on the assignee surviving the assignor, was raised in an originating summons heard by Astbury, J. The case is *In re Williams, Williams* v. *Ball* and is reported (1916) W.N. 250.

The facts are as follows: A testator effected in 1913 a policy of assurance on his life, and handed it over to the defendant, who was his housekeeper, telling her it was for her

to put by; subsequently she told him she could not receive the policy moneys, and he then endorsed the policy as follows: "I authorize Ada Maud Ball, my housekeeper, and no other "person, to draw this insurance in the event of my predeceasing "her, this being my sole desire and intention at the time of "taking this policy out. And this is my signature"; he then signed his name over a penny stamp. No notice of such attempted assignment was given to the insurance company, but it was admitted that notice could be given at any time. The assignor died on 16 February 1916, having by his will appointed the plaintiff his executrix and made certain dispositions of his property. On 10 April 1916, the plaintiff issued this summons to determine (*inter alia*) whether the policy was validly assigned.

By Section 25, sub-section 6, of the Judicature Act, 1873, any absolute assignment (not purporting to be by way of charge only) by an assignor of a debt or chose in action was made sufficient to transfer the legal right. By the Policies of Assurance Act, 1867, assignees of life policies were enabled to sue in their own names, but the assignment was to be in the words or to the effect of the form in the schedule. This form includes the words "in consideration of."

Astbury, J., held that this was not a valid assignment, there being no present words of gift, and the transfer being conditional on the defendant surviving the testator; and further that no consideration was given or expressed to be given; and, therefore, on these grounds, no right of action passed under the Policies of Assurance Act, 1867, or under Section 25, sub-section (6), of the Judicature Act, 1873. The endorsement of the policy only amounted to an equitable attempt by the testator to make an informal gift in his lifetime, and, there being no consideration, it was not operative against his estate.

Bequest of life annuity. Basis of calculation of capital value.

In the case of *In re Castle, Nesbitt* v. *Baugh* (*Law Times*, 13 May 1916) the sole executor of a will asked whether upon the true construction of the will the defendant was entitled to the capital value of the life annuity given to her by the will according to the Government rates or the rates given by an insurance company. By the will dated 19 October 1913, an annuity for £250 free of legacy duty, was bequeathed to the defendant for her life, and the defendant electing to take the capital value of the annuity, which by law

she was entitled to do (*J.I.A.*, vol. xli, p. 427), claimed payment of the value at the date of the testator's death, *i.e.*, 11 May 1914, she being then of the age of 34 years. According to information from the National Debt Office, the cost in 1915 of an annuity on the life of a female aged 35 last birthday was £4,603 5*s.* 11*d.*, the corresponding figure in 1914, when the price of Consols was much higher, being £5,203 11*s.* 2*d.* On the basis of the rates of a Life Assurance Company whose rates had been unaltered since the date of testator's death, the cost would have been £4,412 10*s.* at the date of the testator's death, and £4,365 in 1915.

It appeared that the Government rates for single life annuities under Statutes 10 Geo. 4 c. 24, and 51 and 52 Vict. c. 15 were in some cases less and in others greater than these given in the annuity tables of insurance companies.

Eve, J., held that according to the rule in these cases the annuitant was entitled to the sum which at the testator's death would have purchased a Government annuity of £250 for her life with interest thereon at 4 per-cent.

Are premiums returnable on a policy void for want of insurable interest?

The decision of the Divisional Court in the case of *Hughes* v. *Liverpool Victoria Legal Friendly Society*, which deals with an application for the return of premiums on a policy void for want of insurable interest and which was mentioned in these Notes (*J.I.A.*, vol. l, p. 39) has now been reversed by the Court of Appeal. The case is reported T.L.R.

It will be remembered that Scrutton, J., in the Divisional Court held that the Court ought not to enforce a claim for return of premiums even though obtained by fraud now that the issue of policies on a life in which the insurer has no insurable interest is penalised by statute.

The Court of Appeal held, however, that if an act is prohibited by law, it makes no difference whether it is prohibited under a penalty or left as an indictable misdemeanour, and that the prohibition in 14 Geo. 3 c. 48 (The Gambling Act) was as binding as that of the Assurance Companies Act, 1909.

In giving judgment the Court reviewed the cases of *British Workman's and General Assurance Co., Ltd.*, v. *Cunliffe*, 18 T.L.R. 502 and *Harse* v. *Pearl Life Assurance Co.*, 1904, 1 K.B. 558, and while affirming that where the parties were in *pari delicto* no recovery could be obtained, were of opinion that when

fraud is proved the case cannot be considered as one of *par delictum* and that an innocent plaintiff is entitled to recover.

Courts (Emergency Powers) Acts, 1916.

(*a*) Mortgagee in possession.

The decisions in two cases recently reported in these Notes have now been superseded by the Courts (Emergency Powers) (No. 2) Act, 1916.

The first case is that of *Ziman* v. *Komata Reefs Gold Mining Company, Ltd.* (*J.I.A.*, vol. xlix, p. 287), in which the Court of Appeal held that the expression "mortgagee in possession" applied equally to a mortgagee in possession of personal property as to one in possession of realty.

Section 1, sub-section 1 (*c*) of the Act now provides that the expression "mortgagee in possession" shall not include a mortgagee in possession of property other than land or some interest in land, except where the power of sale had arisen and notice of intended sale had been given prior to 4 August 1914.

(*b*) Foreclosure.

The second case is that of *In re Farnol Eades Irvine & Co., Ltd.* (*J.I.A.*, vol. l, p. 44), in which Warrington, J., held that the prohibition in the Courts (Emergency Powers) Act, 1914, with regard to foreclosure on a mortgage did not prevent a mortgagee from commencing proceedings without leave of Court. Section 1, sub-section 1 (*b*) now extends the provision of the principal Act to the institution of proceedings for foreclosure or sale in lieu of foreclosure.

The Courts (Emergency Powers) (Amendment) Act, 1916. extends the provisions of sub-section 1 of the principal Act (see *J.I.A.*, vol. xlviii, p. 421) so far as officers and men of His Majesty's forces are concerned, to money due and payable in pursuance of a contract made before 11 April 1916, whether such contract was made before or after 4 August 1914. The discretionary powers of the Court may also be exercised even though inability to pay be not caused by circumstances attributable to the war.

Finance Act, 1916. Further limitation of relief from income tax on life assurance premiums.

A further restriction of the relief from Income Tax in respect of life assurance premiums given by the Income Tax Act, 1853, has been made by the Finance Act, 1916. It will be remembered that the Finance Act, 1915, limited the relief given in various directions (see *J.I.A.*, vol. xlix, p. 366) but the increase in the rate of tax still made it possible for insurance and annuity contracts to be

designed and granted solely for the purpose of taking advantage of the tax abatement.

It has therefore been enacted that relief from tax in respect of insurances and contracts for deferred annuities made after 22 June 1916 shall not be given at a greater rate than 3*s*. in the £, and moreover such relief in respect of such insurances made after that date shall not be given, except in respect of premiums or other payments payable on policies for securing a capital sum at death whether in conjunction with any other benefit or not. It is further provided that no relief shall be given in respect of premiums payable during the deferred period on any contract of deferred assurance made after 22 June 1916.

The law as regards the abatement of tax in respect of life assurance premiums now stands as follows:

The limitation, with regard to the maximum amount which may be claimed, to one-sixth of total taxable income remains unaltered, except that the right to claim relief on the basis of the income for the year ending April 1914 is now given (see *J.I.A.*, vol. l, p. 46); the limitations introduced by the Finance Act, 1915, restricting the amount of premiums in respect of which rebate may be claimed (*a*) to 7 per-cent of the actual capital sum assured if the policy secures a capital sum payable at death (whether in conjunction with other benefits or not), or (*b*) to £100 in all if they are payable for any other benefit, also apply to all contracts whenever issued.

In addition to these limitations, policies issued after 22 June 1916 are subject to further regulations as follows:

No rebate will be given unless the policy secure a capital sum payable at death, and if allowed such rebate will not be at a higher rate than 3*s*. in the £.

No rebate will be given in respect of contracts for pure endowments or deferred annuities, or for premiums payable during period of deferment on deferred annuities, provision being made to protect superannuation or *bonâ fide* pension schemes for employees.

As regards super tax the Act is made retrospective, and no relief from the tax can now be claimed in respect of insurance premiums, it being expressly enacted that sub-section 2 (*b*) of Section 66 of the Finance Act, 1910, which gave the right of deducting premiums before arriving at the taxable income, is not now operative.

The Finance Act, 1916, also deals with other questions of interest to life insurance officials.

Section 27 deals with the additional income tax on securities which the Treasury are willing to purchase in connection with the scheme for regulating the foreign exchanges.

Section 37 defines the expression "war insurance premiums" as any additional premium paid in order to extend an existing life assurance to risks arising from war or war service abroad, or any part of a premium paid on a policy covering these risks which appears to the Commissioners to be attributable to the risks. Such payments are entitled to relief from tax and are not subject to the limitations as to one-sixth of income, or the limits of 7 per-cent or one hundred pounds ordinarily applicable to life insurance premiums.

Section 44 (2) provides that income arising from securities issued under Section 47 of the Finance (No. 2) Act, 1915 (which confers power upon the Treasury to issue securities free from tax), shall not be liable to income tax whether remitted to the United Kingdom or not, provided such securities form part of the investments of the foreign life assurance fund of an assurance company.

ACTUARIAL NOTES.

On the determination, by means of Bond-value Tables, of the rate yielded by a redeemable bond when Income Tax is taken into account.

[Continued from *J.I.A.*, vol. xlix, p. 369.]

The approximate formula (6), which is very convenient in practice, may be obtained in the following alternative way, leading also to an approximate limit to the error involved in the formula.

The yield without allowance for tax being J, we have

$$J = g - K/a^{J} = g - K/n - \theta JK$$

$$\therefore \qquad J(1+\theta K) = g - K/n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Similarly,

$$j(1-t) = g(1-t) - K/a^{j(1-t)} = g(1-t) - K/n - Kj(1-t)\theta'$$

$$\therefore \qquad j = g - \frac{K/n}{1-t} - Kj\theta'$$

$$\text{and} \qquad j(1+\theta' K) = g - \frac{K/n}{1-t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\therefore \qquad \frac{j}{J}=\frac{g-\dfrac{K/n}{1-t}}{g-K/n}\cdot\frac{1+\theta K}{1+\theta' K}$$

$$=\left[1-\frac{tK/(1-t)}{gn-K}\right]\left[\frac{1+\theta K}{1+\theta' K}\right] \quad . \quad . \quad (9)$$

Now θ varies slowly, and assuming $\theta=\theta'$ the second factor is unity, and we have formula (6), namely,

$$j=J-\frac{t}{1-t}\cdot\frac{KJ}{gn-K}$$

where it must be remembered that K is negative when the bond is bought at a *discount*.

By means of (9) an approximate limit may be found for the error involved in formula (6). We have

$$(1+\theta K)/(1+\theta' K)=1+\frac{K}{1+\theta' K}(\theta-\theta')$$

$$=1+\frac{K}{1+\theta' K}[J-j(1-t)]\frac{d\theta}{dJ}, \text{ nearly} \quad . \quad (10)$$

From formula (6) we have approximately

$$J-j(1-t)=J-(1-t)J+tKJ/(gn-K), \text{ nearly}$$

$$=tJ[1+K/(gn-K)] \qquad ,,$$

$$=tJ\frac{gn}{gn-K}=tJ\frac{g/J}{1+\theta K} \qquad ,, \qquad \text{from (7)}$$

$$=tg/(1+\theta K) \qquad ,,$$

$\therefore$ from (10)

$$\frac{1+\theta K}{1+\theta' K}=1+\frac{tKg}{(1+\theta K)(1+\theta' K)}\cdot\frac{d\theta}{dJ}$$

or putting the near value, ·6 for θ and θ'

$$1+\frac{tKg}{1+1{\cdot}2K}\cdot\frac{d\theta}{dJ}, \text{ nearly.}$$

Hence from (9) the error is approximately

$$\frac{tKg}{1+1{\cdot}2K}\cdot J\frac{d\theta}{dJ}$$

Now it will be found (see Table appended)* that for

* *Cf.* the algebraical value obtainable from equation (18) *T.B.*, Part II, p. 433, in which J must be substituted for x and n for t.

practical rates and periods $J\frac{d\theta}{dJ}$ may be regarded as a function of (nJ); also that it never exceeds $\frac{nJ}{12}$, which is a fairly close approximation when n is not great, nor exceeds a numerical value of about $\frac{1}{6}$. Hence the error is

$$< \frac{tKgnJ}{12(1+1\cdot 2K)}, \text{ approximately}$$

and

$$< \frac{tKg}{6(1+1\cdot 2K)} \quad \text{,,}$$

the first expression giving the closer limit if $nJ<2$.

Applying the former expression to the second example given, *J.I.A.*, vol. xlix, pp. 368–9, we find that the limit of error in that case is

$$\frac{\cdot 1 \times \cdot 236 \times \cdot 07 \times 20 \times \cdot 051}{12 \times 1\cdot 28} = \cdot 00011 \text{ or } \cdot 011 \text{ per-cent},$$

the actual error being ·010 per-cent. The error may, however, be much larger if a high value of K be combined with a high rate of interest, a high rate of tax and a long term, in which case, however, a high degree of accuracy is impracticable, even with an "exact" formula, owing to the impossibility of forecasting an average value of t, the rate of tax.

G. J. L.

TABLE

Showing the values of $F = J\frac{d\theta}{dJ}$, *where* $\theta = \frac{1}{J}[1/a_{\overline{n}|} - 1/n]$.

nJ	$nJ/12$	J = ·03		J = ·04		J = ·06	
		n	F	n	F	n	F
·30	·025	10	·024	...	...	...	...
·40	·033	...	...	10	·032	...	...
·60	·050	20	·048	15	·047	10	·046
·80	·067	...	...	20	·062	...	...
·90	·075	30	·070	...	...	15	·068
1·20	·100	40	·091	30	·090	20	·088
1·50	·125	50	·109	...	...	25	·106
1·60	·133	...	...	40	·114	...	...
1·80	·150	60	·125	45	·124	30	·122
2·00	·167	...	...	50	·134	...	...
2·40	·200	80	·150	60	·149	40	·146
3·00	·250	100	·165	75	·163	50	·162
3·60	·300	120	·171	90	·170	60	·168
4·20	·350	140	·171	105	·170	70	·168

Ordinary and Osculatory Interpolation.

In comparisons of the ordinary and the osculatory methods of interpolation a single ordinary formula is always used for all the values to be interpolated between a pair of given values. This is not necessary, and does not do justice to the possibilities of the ordinary method. There is nothing in the principles of interpolation requiring that each space should be dealt with separately as if the values to be inserted in it were not connected with those inserted in the neighbouring spaces. In practice, if it is necessary to complete a table of rates of which every fifth or tenth value has been calculated, a course sometimes adopted, if the differences are too large for the spaces to be filled up by inspection, is to insert one or two values in each space by interpolation, and then to fill up the intervals suitably with reference to the neighbouring values—calculated and interpolated—on both sides. The object of this Note is to consider the systematic application of this process.

Suppose that the given values are . . . u_{-5}, u_0, u_5 . . . , and that it is required to insert 4 values at equal intervals between each pair. Then, if the differences of the given values are such as to admit of interpolation by differences, we may, among various possible courses :

(i) interpolate u_{-1}, u_1 from $u_{-10} \ldots u_{10}$, and u_4, u_6 from $u_{-5} \ldots u_{15}$, by fourth differences, and then insert u_2, u_3 by interpolation between u_{-1}, u_0, u_1 and u_4, u_5, u_6;

or (ii) interpolate u_{-3}, u_{-2} from $u_{-10} \ldots u_5$, and u_2, u_3 from $u_{-5} \ldots u_{10}$, by third differences, and then insert u_{-1}, u_1 by interpolation between u_{-3}, u_{-2}, u_0, u_2 and u_3;

or (iii) if the retention of the given values is not material, interpolate u_{-3}, u_{-2}, &c., as in (ii), and then insert u_{-1}, u_0, u_1 by interpolation between u_{-3}, u_{-2} and u_2, u_3.

The first of these processes is closely analogous to the fifth difference osculatory method, the only difference being that in the latter we determine $u_{-\theta}$, u_θ, $u_{5-\theta}$ and $u_{5+\theta}$ (where θ is indefinitely small) from the given values, and then insert u_1, u_2, u_3, u_4 by interpolation between $u_{-\theta}$, u_0, u_θ, and $u_{5-\theta}$, u_5, $u_{5+\theta}$.

The second process is similarly analogous to the osculatory process represented by the formula

$$u_x = u_0 + xa_0 + \frac{x^2}{2}b_0 + \frac{x(x^2-1)}{6}c_0 + \frac{x^2(5x^2-2\frac{3}{4})}{12}d_0$$

which gives the values of u between $u_{-\frac{1}{2}}$ and $u_{\frac{1}{2}}$ on the basis of the successive partial interpolation curves meeting and having the same slope in the middles of the spaces to be filled up.

The relative accuracy of these interpolation processes and the osculatory processes will depend to some extent, in any particular case, on the nature of the differences of the given values; but it would appear that in general (i) will be slightly less accurate than the fifth difference osculatory process, since the latter takes account of the fifth difference—although to a very small extent—in the values it assigns to u_{-1}, u_1, and that (ii) will be more accurate than the third difference osculatory process, since the interpolated values of u_2, u_3 (based on the same original values as the u_2, u_3 given by the osculatory formula) are correct to third differences. Taking, as an example, the *Text-Book* values of q for ages 45 to 50, as obtained to six places from the table of log p on p. 88), we find that (i) and the corresponding osculatory process both give the intervening values correctly to the sixth place. The results given by (ii) and the third difference osculatory formula are as follows:

Age	True Value	Value by (ii)	Third Difference Osculatory Value
46	·012818	·012817	·012825
47	·013451	·013448	·013452
48	·014144	·014141	·014136
49	·014903	·014902	·014892

The question of accuracy is, however, less material to the present investigation than that of smoothness, for the osculatory method was devised to smooth away the irregularities resulting from ordinary interpolation. In the osculatory process the successive partial interpolation curves have single or double contact at their points of junction, that is to say, they have two or three coincident points in common. In the suggested ordinary interpolation processes they overlap, having two or three points of the interpolated curve in common, and we have to see whether this has a similar effect in producing smoothness.

For purposes of comparison we may take a section of the q column of the English Life Table No. 8. It will be sufficient to take the third difference osculatory values as a standard, since it appears from Mr. King's investigations (*J.I.A.*, vol. xliii. p. 136) that there is little to choose as regards smoothness between the third and fifth difference values. As colog p throughout the greater part of the table must be approximately of the form $a+bc^x$, it is possible that the best results might be obtained by using log (colog $p-a$), with an average value of a, as a basis of interpolation; but the difference in the results could not apparently be material, and as the official values have been interpolated from log $(\cdot 1+q)$ it will be convenient to work with that function. The resulting values by the three ordinary interpolation processes and the official values by the third difference osculatory process, with their second differences, are shown in the following table:

Age	By Overlapping Interpolation						By Osculatory Interpolation (third difference)	
	Method (i)		Method (ii)		Method (iii)			
	q	$10^6\Delta^2$	q	$10^6\Delta^2$	q	$10^6\Delta^2$	q	$10^6\Delta^2$
55	021118	123	021116	119	021110	125	021111	119
56	022716	142	022716	129	022707	132	022716	123
57	024437	141	024435	146	024429	140	024440	139
58	026300	140	026283	165	026283	138	026287	157
59	028301	144	028277	141	028277	155	028273	177
60	030448	170	030436	146	030409	172	030416	150
61	032736	166	032736	177	032696	191	032736	147
62	035194	210	035182	240	035155	213	035206	210
63	037818	276	037805	245	037805	277	037823	275
64	040652	332	040668	320	040668	305	040650	351
65	043762	338	043776	367	043808	334	043752	363
66	047204	355	047204	386	047253	369	047205	347
67	050984	399	050999	373	051032	406	051021	383
68	055119	454	055180	386	055180	460	055184	421
69	059653	499	059734	514	059734	480	059730	464
70	064641	510	064674	578	064748	503	064697	567
71	070128	569	070128	572	070242	528	070128	605
72	076125	555	076160	476	076239	555	076126	548
73	082691	507	082764	513	082764	508	082729	479
74	089812	469	089844	507	089844	508	089880	397
75	097440	506	097437	501	097432	508	097510	495
76	105537	550	105537	497	105528	500	105537	615
77	114140	503	114138	497	114132	491	114059	539
78	123293	416	123236	536	123236	405	123196	446
79	132949	354	132831	354	132831	411	132872	331
80	143021	400	142962	277	142831	413	142994	293
81	153447	302	153447	335	153242	416	153447	390
82	161273	405	164209	559	164066	416	164193	473

From inspection of the second differences it would appear that the results given by the first and second methods are little, if at all, less smooth than those given by the osculatory method, and that those given by the third method—which involves more graduation—are appreciably smoother. The effects of the various methods are, however, to some extent masked by the use of logarithms, and it may be useful to consider another example in which the interpolation is based directly on the given values. For this purpose we may take a section of the male population table as at the 1911 census. The differences of the populations in the five-year age-groups are of such a nature that the employment of fourth differences would serve no useful purpose. The comparison in the following table is accordingly restricted to the results given by methods (ii) and (iii); in the application of the latter method the populations at ages 28, 29, 33, 34, &c., have been interpolated centrally from the five-year age-groups instead of from the "pivotal" values.

Age	By Overlapping Interpolation				By Osculatory Interpolation (third difference)	
	Method (ii)		Method (iii)			
	Population	Δ^2	Population	Δ^2	Population	Δ^2
28	288191	−224	288165	−119	288165	−204
29	285396	110	285376	171	285423	283
30	282377	110	282468	223	282477	124
31	279248	217	279389	274	279248	70
32	276009	438	276087	327	275895	310
33	272553	288	272511	719	272472	552
34	268659	917	268608	694	268739	791
35	264477	1109	263986	669	264454	1054
36	259378	867	258670	646	259378	892
37	253170	187	252685	620	253248	502
38	246095	310	246054	131	246226	108
39	238833	+ 32	238803	48	238702	+283
40	231261	206	231421	+ 37	231070	184
41	223721	212	223991	119	223721	− 77
42	216387	50	216598	204	216556	+ 71
43	209265	49	209324	327	209314	221
44	202193	463	202254	310	202143	367
45	195170	586	195511	295	195193	550
46	188610	422	189078	279	188610	450
47	182636	− 33	182940	262	182577	178
48	177084	+153	177081	−213	176994	− 91
49	171499	−403	171484	213	171589	365
50	166067	588	165674	215	166093	540
51	160232	405	159651	214	160232	361
52	153809	+149	153413	216	153831	94
53	146981	− 27	146961	+235	147069	+171
54	140302	+389	140293	257	140213	436
55	133596	544	133860	282	133528	510
56	127279	439	127684	303	127279	334
57	121506	76	121790	327	121540	221

It will be noticed, on inspection of the second differences in cols. (3) and (7), that the waves which constitute a well-known feature of the osculatory method are rather closely imitated by method (ii), but their crests are slightly exaggerated, their troughs apparently somewhat filled up and their sides broken by secondary waves. It may be questioned, incidentally, whether these regular waves are really much less open to objection than the abrupt changes of curvature produced by the old-fashioned interpolation, since both must be equally unknown to the true curve. The second differences by method (iii) appear somewhat more normal, and the changes of curvature —although rather abrupt at two points—will be found to reflect exactly on a greatly reduced scale the changes in the five-year age-groups. Both methods (ii) and (iii) involve a more extensive range than the third difference osculatory method, and consequently produce larger deviations from the populations in the five-year age-groups. This may be a disadvantage in the particular example, but it does not affect the question under consideration.

The foregoing investigation seems to justify the view that ordinary interpolation suitably employed will in general give as good results as the osculatory method, and that in some cases—especially when it is desirable to work as correctly as possible to third differences, and at the same time to obtain a reasonably smooth series of values—it may be preferable. It is perhaps worth while, therefore, to indicate a simple method of performing the special interpolations required in the application of the suggested processes.

If the values u_{-3}, u_{-2}, u_2, u_3, &c., are calculated by ordinary central interpolation we can fill up the intervening spaces as shown by the following numerical illustration :

u	Δu	Differences in A.P.	Correction
209324			
	7070		
202254			
	6743	6775·4	32·2
195511			
	6433	6480·8	48·3
189078			
	6138	6186·2	48·3
182940			
	5859	5891·6	32·2
177081			
	5597		
171484			

The first two and the last two numbers in col. (1) are the calculated values, and the first and last numbers in the next column are their differences. In col. (3) we write down the intervening differences in arithmetical progression. The sum of the numbers in col. (3)—or twice the sum of 7,070 and 5,597—being 25,334, whereas the sum of the actual differences has to be 25,173, we distribute the excess in col. (4 in the proportion of $\frac{2}{10}$, $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{2}{10}$; deducting these adjustments from the numbers in col. (3) we insert in cols. (2) and (1) the actual differences and the required intervening values. It can easily be shown that this has the effect of making the third differences of col. (1) constant.

If the values u_{-1}, u_0, u_1, u_4, u_5, u_6 have been calculated and the interpolated values of u_2, u_3 are required, we proceed as follows:

u	Δu	$\Delta^2 u$	Second Differences in A.P.	Adjustment	Modified Second Differences	Corresponding First Differences	Correction
83209							
	5691						
88900		360					
	6051						
94951			371·2	10·0	361·2		
	6451					6412·2	38·9
101402			382·4	15·0	367·4		
	6838					6779·6	58·3
108240			393·6	15·0	378·6		
	7197					7158·2	38·8
115437			404·8	10·0	394·8		
	7553						
122990		416					
	7969						
130959							

Here the first three and the last three numbers in col. (1) are the calculated values. Dealing with their first differences we proceed as in the previous illustration and obtain in col. (7) the intervening differences on the assumption of constant third differences; the sum of these being 20,350, whereas the sum of the actual differences has to be 20,486, we distribute the defect in col. (8) in the proportion of $\frac{2}{7}$, $\frac{3}{7}$ and $\frac{2}{7}$; adding these adjustments to the numbers in col. (7) we obtain the actual first differences and the two intervening values. The effect of this is to make the fifth differences of col. (1) constant.

Finally, if the values of u_{-3}, u_{-2}, u_0, u_2 and u_3 have been calculated, and the values of u_{-1}, u_1 are required, we proceed as follows:

u	Δu	First Differences in A.P.	First Correction	Modified First Differences	Second Correction
72853					
	4991				
77844					
	5356	5353·8	18·2	5335·6	20·7
83200					
	5700	5716·6	27·3	5689·3	10·4
88900					
	6042	6079·4	27·3	6052·1	10·4
94942					
	6403	6442·2	18·2	6424·0	20·7
101345					
	6805				
108150					

Here the first and last two numbers and the middle numbers in col. (1) are the calculated values. Neglecting the middle numbers for a moment we complete the differences as in the first illustration, the results being shown in col. (5). These would be the differences if the middle value were abandoned. The retention of the middle value makes the sum of the first two differences 31·1 too small, and the sum of the last two 31·1 too great. Distributing the excess and defect in the proportions of $\frac{2}{3}$ and $\frac{1}{3}$, we obtain the adjustments in col. (6) and hence the correct first differences and the intervening values.

With reference to the statement in an earlier paragraph that the populations at ages 28, 29, 33, 34, &c., were interpolated centrally from the five-year age-groups, it may be useful to mention that if $\Sigma_{-7}^{-3} u = w_{-1}$, $\Sigma_{-2}^{2} u = w_0$, &c., the value of u_x in terms of the w's may be written in the Everett form:

$$u_x = \frac{\xi}{25} w_0 + \frac{x}{25} w_1 + \frac{\xi^3 - 31\xi}{3750} \Delta^2 w_{-1} + \frac{x^3 - 31x}{3750} \Delta^2 w_0$$

where ξ is measured backwards from the point 5.

If u_2 and u_3, the middle values of the 20 values u_{-7} to u_{12} are required, this gives the convenient relations

$$u_2 + u_3 = \cdot 2(w_0 + w_1) - \cdot 032(\Delta^2 w_{-1} + \Delta^2 w_0)$$
$$u_2 - u_3 = -\cdot 04 \Delta w_0 - \cdot 0032(\Delta^2 w_{-1} - \Delta^2 w_0).$$

R. T.

A General Method of obtaining Interpolation Formulas.

LET $\ldots u_{-s},\ u_{-(s-1)},\ \ldots u_0 \ldots u_{r-s-1},\ u_{r-s} \ldots$ be values of a function of x corresponding to the values $\ldots a_{-s},\ a_{-(s-1)} \ldots 0 \ldots a_{r-s-1},\ a_{r-s}, \ldots$ of the variable, and let their successive differences be formed by Dr. Sheppard's method (Enc. Brit. *Interpolation*) as shown in the following scheme:

x	u_x	$\Delta' u_x$	$\Delta'^2 u_x$	$\Delta'^3 u_x$
⋮	⋮			
a_{-s}	u_{-s}	⋮		
		$(u_{-(s-1)}-u_{-s})/(a_{-(s-1)}-a_{-s})$		
$a_{-(s-1)}$	$u_{-(s-1)}$	⋮	⋮	
⋮	⋮	⋮	⋮	⋮
a_{-1}	u_{-1}			
		$(u_0-u_{-1})/(-a_{-1})$		
0	u_0		$(\Delta' u_0-\Delta' u_{-1})/\frac{1}{2}(a_1-a_{-1})$	
		$(u_1-u_0)/a_1$		$(\Delta'^2 u_0-\Delta'^2 u_{-1})/\frac{1}{3}(a_2-a_{-}$
a_1	u_1		$(\Delta' u_1-\Delta' u_0)/\frac{1}{2}a_2$	
		$(u_2-u_1)/(a_2-a_1)$		
a_2	u_2	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
a_{r-s-1}	u_{r-s-1}			
		$(u_{r-s}-u_{r-s-1})/(a_{r-s}-a_{r-s-1})$		
a_{r-s}	u_{r-s}	⋮		
⋮	⋮	⋮		

where, generally,

$$\Delta'^r u_s = (\Delta'^{r-1} u_{s+1} - \Delta'^{r-1} u_s) \Big/ \frac{1}{r}(a_{s+r}-a_s)*$$

It will be readily seen from the scheme that if $u_{-s}, u_{-(s-1)} \ldots u_{r-s-1}$ are all zero

$$\Delta'^r u_{-s} = \frac{u_{r-s}}{(a_{r-s}-a_{r-s-1}) \cdot \frac{1}{2}(a_{r-s}-a_{r-s-2}) \ldots \frac{1}{r}(a_{r-s}-a_{-s})}$$

$$= \frac{r!\ u_{r-s}}{(a_{r-s}-a_{r-s-1})(a_{r-s}-a_{r-s-2}) \ldots (a_{r-s}-a_{-s})}$$

* In using the symbol Δ' the writer of the Note has followed the late Mr. T. G. Ackland (*J.I.A.*, vol. xlix, p. 369). This symbol has been used for other purposes (see *e.g.* Enc. Brit., *Calculus of Finite Differences*), but as it may be regarded as a symbol of a general nature, denoting a difference but not the ordinary difference, it is thought that no confusion can arise. If a distinctive symbol were required, an italic Δ might be appropriate, since these divided differences in which the divisor is $\frac{1}{r}(a_{s+r}-a_s)$ become the ordinary differences if the intervals are made equal, and the inequality of the sides of the italic symbol would suggest the inequality of the intervals. The difference for which Dr. Sheppard uses the symbol ∇ elsewhere in this issue is the Newtonian difference, in which the divisor is $(a_{s+r}-a_s)$ instead of $\frac{1}{r}(a_{s+r}-a_s)$. In De Morgan's *Differential and Integral Calculus* this difference is denoted by θ, but this symbol is now frequently used for other purposes in connection with Interpolation.—[EDS. *J.I.A.*]

and, similarly, that if $u_{-(s-1)} \ldots u_{r-s}$ are all zero

$$\Delta'^r u_{-s} = \frac{(-1)^r r!\ u_{-s}}{(a_{-(s-1)}-a_{-s})(a_{-(s-2)}-a_{-s}) \ldots (a_{r-s}-a_{-s})}$$

It may be observed also that if u_x is of the form kx^r

$$\Delta' u_o = ka_1^{r-1};\ \Delta' u_1 = k(a_2^{r-1} + a_2^{r-2}a_1 + \ldots + a_1^{r-1});\ \&c.,$$

i.e., Δ' is of the $(r-1)$th degree in a. Writing $\Delta' u_1$ and $\Delta' u_2$ in the form $\Sigma a_2^\theta a_1^\phi$ and $\Sigma a_3^\theta a_2^\phi$, where θ and ϕ have all values from 0 to $r-1$ subject to the condition $\theta+\phi=r-1$, we see that $\Delta' u_2 - \Delta' u_1 = \Sigma a_2^\phi(a_3^\theta - a_1^\theta)$ and hence, dividing by $\frac{1}{2}(a_3-a_1)$, that $\Delta'^2 u_1 = 1\,.\,2\,.\,k\Sigma a_3^\theta a_2^\phi a_1^\psi$, where θ, ϕ, ψ have all values from 0 to $r-2$ subject to the condition $\theta+\phi+\psi=r-2$, *i.e.*, Δ'^2 is of the $(r-2)$th degree in a. Proceeding in this way, we shall find that each difference is one degree lower in a than the preceding one. It follows that if u_x is a rational integral function of x of the rth degree all differences above the rth vanish, when the differences are formed as above.

In interpolating u_x by Finite Differences from given values of u we regard the u's as the ordinates of a curve represented by a rational integral function of x, and we take u_x to be the ordinate at the point x of the curve determined by the given values. It follows that if we base our interpolation on $r+1$ u's, u_x will be a rational integral function of x of a degree not exceeding r, for from the $r+1$ given values we can determine $r+1$ constants; and the coefficients of the successive powers of x will be linear compounds of the $r+1$ given u's. By rearrangement of terms u_x can be expressed as a linear function of the $r+1$ u's with coefficients which will be rational integral functions of x of degree not exceeding r.

Referring now to the scheme given above, any $r+1$ consecutive u's can be expressed in terms of some one of them, u_0 say, and of any r successive differences, subject only to each difference descending or ascending by one step from the difference immediately preceding it (or, as regards the first difference, from u_0) and keeping within the triangle formed by the $r+1$ u's and their differences; for each such difference involves one of the $r+1$ u's not involved in preceding differences (as will be obvious on inspection of the scheme), and the r differences expressed in terms of the u's will

consequently give r independent equations from which we can determine the u's other than u_0 as linear compounds of u_0 and the r differences.

Hence u_x, if interpolated from $r+1$ consecutive u's of which u_0 is one, can be expressed as a linear function of u_0 and r successive differences as above, and the coefficients will be rational integral functions of x of a degree not exceeding r.

Let the $r+1$ u's be $u_{-s} \ldots u_{r-s}$, so that the last term in the expression for u_x is $A_r \Delta'^r u_{-s}$ where A_r is a rational integral function of x of a degree not exceeding r. Since A_r is independent of the u's we may, in order to determine it, assign any values we please to the u's.

The u involved in $\Delta'^r u_{-s}$ and not involved in the preceding differences will be u_{r-s} or u_{-s} according as the preceding difference is $\Delta'^{r-1} u_{-s}$ or $\Delta'^{r-1} u_{-(s-1)}$.

If the preceding difference is $\Delta'^{r-1} u_{-s}$ let us assign to each of the u's from u_{-s} to u_{r-s-1} the value zero. Then in the expression for u_x all the terms vanish except $A_r \Delta'^r u_{-s}$; and (since u_x is the ordinate at the point x of the curve determined by $u_{-s} \ldots u_{r-s}$) this must vanish when x has any of the r values a_{-s}, $a_{-(s-1)} \ldots a_{r-s-1}$, and it must become u_{r-s} when x has the value a_{r-s}. Hence, since $\Delta'^r u_{-s} = r!\, u_{r-s}/(a_{r-s}-a_{r-s-1}) \ldots (a_{r-s}-a_{-s})$ when $u_{-s} \ldots u_{r-s-1}$ are all zero,

$$A_r = \frac{(x-a_{-s})(x-a_{-(s-1)}) \ldots (x-a_{r-s-1})}{r!}$$

Similarly, if the preceding difference is $\Delta'^{r-1} u_{-(s-1)}$, we may assign the value zero to each of the u's from $u_{-(s-1)}$ to u_{r-s}, and proceeding as before we shall find that

$$A_r = \frac{(x-a_{-(s-1)})(x-a_{-(s-2)}) \ldots (x-a_{r-s})}{r!}$$

Although we have found the coefficient of the last term only in the expression for u_x as interpolated from $r+1$ u's, the form of the result is general. For suppose that we have to interpolate u_x from any number of consecutive u's (including $u_{-s} \ldots u_{r-s}$) and that $\Delta'^r u_{-s}$ is one of the differences in the expression for u_x, so that all the preceding differences are within the triangle formed by $u_{-s} \ldots u_{r-s}$ and their differences. Then, in accordance with the principle that any convenient values may be assigned to the u's for the purpose of determining the coefficients, all the u's outside the

range $u_{-s} \ldots u_{r-s}$ may be put equal to the corresponding ordinates of the curve determined by $u_{-s} \ldots u_{r-s}$. The expression for u_x must then become exactly the same as if it were determined from $u_{-s} \ldots u_{r-s}$. It will consequently become of the rth degree in x, all differences above the rth will vanish, and the coefficient of $\Delta'^r u_{-s}$ will be A_r as found above.

Having shown that u_x can be expressed as a linear function of u_0 and any sequence of differences, descending, ascending or zigzagging, subject as above, and having found the general form of the coefficient of any difference in the sequence according as it is a descending or an ascending difference, we can write down any formula of the class under consideration.

Or we can deduce the principles of formation enunciated by Dr. Sheppard in the article already referred to. These principles form such convenient rules for writing down interpolation formulas that it may be useful to reproduce them here:

"(i) We start with any tabulated value of u,

(ii) we pass to the successive differences by steps, each of which may be either downwards or upwards, and

(iii) the new suffix which is introduced at each step determines the new factor (involving x) for use in the next term."

In order to obtain an interpolation formula in the Everett form when the intervals are unequal we must write $\xi = a_1 - x$, $\alpha_1 = a_1$, $\alpha_2 = a_1 - a_{-1}$, &c., $\alpha_{-1} = a_1 - a_2$, &c., so that ξ and the α's are measured from a_1 just as x and the a's are measured from 0 but in the reverse direction. By a method similar to that followed in the discussion of the general case we can show that u_x may be expressed as a linear function of u_0, u_1 and the even differences $\Delta'^2 u_{-1}$, $\Delta'^2 u_0 \ldots \Delta'^{2r} u_{-r}$, $\Delta'^{2r} u_{-r-1} \ldots$, and that the coefficients of the rth pair of differences will be rational integral functions of x of degree not exceeding $2r+1$. To determine the coefficient of $\Delta'^{2r} u_{-r-1}$ (B_r, say) we assign the value zero to each of the u's from u_{-r} to u_r, and we put all the u's outside the range $u_{-r} \ldots u_{r+1}$ equal to the corresponding ordinates of the curve determined by $u_{-r} \ldots u_{r+1}$. All the terms then vanish except $B_r \Delta'^{2r} u_{-(r-1)}$, and this must vanish when x has any of the values $a_{-r} \ldots a_r$ and must become u_{r+1} when x has the value

a_{r+1}. Hence, since $\Delta'^{2r}u_{-(r-1)} = (2r)!\, u_{r+1}/(a_{r+1}-a_r) \dots (a_{r+1}-a_{-(r-1)})$ when $u_{-r} \dots u_r$ are all zero,

$$B_r = \frac{(x-a_{-r})(x-a_{-(r-1)}) \dots (x-a_r)}{(2r)!\,(a_{r+1}-a_{-r})}$$

From considerations of symmetry the coefficient of $\Delta'^{2r}u_{-r}$ must be identically the same function of ξ and the a's. Hence

$$u_x = \frac{\xi}{a_1}u_0 + \frac{(\xi-a_{-1})\xi(\xi-a_1)}{2!\,(a_2-a_{-1})}\Delta'^2u_{-1} + \dots$$

$$+ \frac{x}{a_1}u_1 + \frac{(x-a_{-1})x(x-a_1)}{2!\,(a_2-a_{-1})}\Delta'^2u_0 + \dots$$

This result may of course be obtained from the ordinary formula in terms of u_0, $\Delta'u_0$, Δ'^2u_{-1}, $\Delta'^3u_{-1} \dots$ by expressing each odd difference in terms of the preceding pair of even differences.

R. T.

Notes on the Mortality Experience of Missionaries of the Wesleyan Methodist Missionary Society.

THE following notes have been compiled from material supplied by the Society, and are published here by permission.

The paucity of the data prevents the formation of any rates of mortality, but it is hoped that some indications regarding the important question of the risks of foreign residence may be gleaned from the figures. In this connection it has to be borne in mind that the Society is one of the largest of the British Missionary Institutions, and that the experience extends over 20 years.

(1) The data consist of male British lives.

(2) The period covered by the observations extends from 1 September 1895 to 31 August 1915.

(3) Each case was observed from 1 September 1895 (or later entry into the service) until 31 August 1915 (or earlier exit).

(4) For the particular purpose for which the investigation was made it was desired to approximate as closely as possible to the net time spent upon the foreign field, and the following proportions were deducted from the gross total exposures in each age group on account of furlough:

For West Africa—one-quarter.
For other stations—one-seventh.

(These proportions were suggested by the Officers of the Society. It should, perhaps, be added that the data for each man comprised merely the year of commencement of foreign service and the year of final cessation of such service.)

(5) The data were divided into five broad groups upon a geographical basis as follows:

India, Burma, Ceylon.
China.
West Indies.
West Africa.
South Africa.

It was not feasible to make any further division.

The following brief statements show where the lives were chiefly concentrated:

India, &c.—In India, Burma and Ceylon the stations are very widely spread. The chief districts *not* represented are the Central Provinces, the Central Indian Agencies and Rajputana, also outlying districts such as Kashmir, Nepal and Northern Burma.

China.—The stations lie roughly along a line drawn from Hongkong through Canton and the Hunan Province to the Yangtse River Valley round Hankow and Wuchang.

West Indies.—Practically all of the islands are covered except Cuba. The section also includes the eastern coast of Central America, the Panama Canal zone and British Guiana.

West Africa.—The chief districts are Southern Nigeria, the Gold Coast Colony and Sierra Leone. There are also stations in Ashanti and in Gambia.

South Africa.—With one exception the stations are all south of the Zambesi River. They are fairly widely spread, the chief concentrations being in the Transvaal and in Southern Rhodesia.

(6) The following summary gives the results of the tabulation together with the number of deaths expected according to a table representing the mortality of a similar class of lives at home and the excess of the actual over the expected deaths. This table has not been published, and it was only made available to the author by the courtesy of the Actuaries who compiled it. The chief feature of the table is the extremely light mortality, the values of q_x for example, at ages 30, 50 and 70 being practically 40 per-cent of those by the O^M

Experience. It is obvious that the standard is a very severe one—indeed, for all ages combined, there would appear to be little evidence of any extra mortality (save, perhaps, for West Africa) if the O^{M} Table be taken as a basis.

	India, &c. (254 *cases*).				*China* (67 *cases*).			
Age Group	*Years of Exposure	Actual Deaths	Expected Deaths	Δ	*Years of Exposure	Actual Deaths	Expected Deaths	Δ
–24	176	...	·4	–·4	49	...	·1	–·1
25–29	595	4	1·2	2·8	151	...	·3	–·3
30–34	491	2	1·2	·8	131	2	·3	1·7
35–39	316	3	1·0	2·0	92	1	·3	·7
40–49	251	1	1·1	–·1	114	2	·5	1·5
50–59	77	1	·6	·4	31	...	·3	–·3
60–	14	...	·2	–·2	...	...	...	...
All Ages	1,920	11	5·7	5·3	568	5	1·8	3·2

	West Indies (112 *cases*).				*South Africa* (110 *cases*).			
Age Group	*Years of Exposure	Actual Deaths	Expected Deaths	Δ	*Years of Exposure	Actual Deaths	Expected Deaths	Δ
–24	53	...	·1	–·1	71	...	·1	–·1
25–29	270	2	·6	1·4	303	1	·6	·4
30–34	137	1	·4	·6	231	1	·6	·4
35–39	33	..	·1	–·1	117	...	·4	–·4
40–49	40	...	·2	–·2	82	...	·4	–·4
50–59	18	...	·1	–·1	28	...	·2	–·2
60–	9	1	·2	·8	16	1	·3	·7
All Ages	560	4	1·7	2·3	848	3	2·6	·4

	West Africa (51 *cases*).				*All* (*ex. West Africa*) (543 *cases*).				*All* (594 *cases*).			
Age Group	*Years of Exposure	Actual Deaths	Expected Deaths	Δ	*Years of Exposure	Actual Deaths	Expected Deaths	Δ	*Years of Exposure	Actual Deaths	Expected Deaths	Δ
–24	34	...	·1	–·1	349	...	·7	–·7	383	...	·8	–
25–29	92	2	·2	1·8	1,319	7	2·7	4·3	1,411	9	2·9	6
30–34	53	2	·1	1·9	990	6	2·5	3·5	1,043	8	2·6	5
35–39	16	...	·1	–·1	558	4	1·8	2·2	574	4	1·9	2
40–49	5	...	...	...	487	3	2·2	·8	492	3	2·2	
50–59	14	...	·1	–·1	154	1	1·2	–·2	168	1	1·3	–
60–	7	...	·1	–·1	39	2	·7	1·3	46	2	·8	1
All Ages	221	4	·7	3·3	3,896	23	11·8	11·2	4,117	27	12·5	14

* N.B.—See (4) *re* deduction made for furlough. These figures are net.

(7) It may be of interest to add that it was found possible to trace the history of those who returned from the mission field and that their experience to 31 August 1915 appears strongly to indicate that no excessive mortality occurs after such return. The men enter, with hardly an exception, the ministry of the Wesleyan Methodist Church in this country, and so can be traced. The few who do not enter the home ministry are, of course, hard to locate, and in the present instance it was thought sufficient to enquire of the Missionary Authorities as to the reasons for their leaving the foreign field. Only in one case was ill-health the cause, and the man affected died soon after exit. His death has been taken into account in arriving at the conclusion above-mentioned.

(8) It is important to add that the Society not only exercises the greatest care in selecting its candidates, but that it also insists upon the observance, during residence abroad, of certain rules regarding health. Moreover, the men come frequently before the doctor in connection with furlough, &c. It is, therefore, safe to say that the lives in question are of a particularly select class.

(9) The number of missionaries employed by the Society increased very considerably during the period. It would doubtless have been interesting to have an analysis of the experience according to years of service, but it is thought that such an analysis would not be of much value in this case, owing to the frequent periods of furlough with consequent re-examination. Accordingly the question has not been investigated.

(10) It must remain a matter for sincere regret that the figures are so very scanty. Taken by themselves they can only be regarded as indicative of general tendencies. If, however, it were possible to secure returns from similar societies for the same period the combined data might yield important results. In order that no chance of achieving this end may be lost, the data for the Wesleyan Methodist Missionary Society are now placed upon record.

R. C. S.

REVIEWS.

Royal Commission on Venereal Diseases. Final Report [Cd. 8189]. *Minutes of Evidence, &c.* [Cd. 7474 and 8189.]

AN actuary is interested in the study of disease because it affects the health, longevity and fertility of the community, and if he had to examine the effect of a particular disease he would endeavour to obtain data from which he could ascertain the proportion of

the general population suffering from the disease, the proportion of the general population that has suffered from it, and the mortality and sickness experienced subsequent to the disease and the number of children born to diseased members and the health of the children. The investigation would be made for small groups of ages, and for either sex, and would divide the data to show whether occupation, social status or residence was an important factor; the results in every investigation would be shown side by side with corresponding figures for the general population. The investigations would be made over a number of years, and would, therefore, show whether the disease is increasing. It is clear that with such information we should have full knowledge of the prevalence and effects of a disease, and if samples of the diseased population are given different treatment we could even measure the success of the various attempts to cure the disease. Unfortunately, however, suitable data can seldom be found: they do not exist in our population statistics, and there is great difficulty in obtaining smaller samples of the general population suitable for the purpose, so that we are tempted to lose sight of the real problem, or at the best treat it as an ideal impossible to attain. This may be an excuse for many weak investigations, but however bad our data we must bear in mind what we ought to require to the extent of remembering that the effects of a disease are not measured by the "possible effects" in a few cases, but by (1) the average amount of sickness of all kinds that is subsequently suffered by the "diseased", and (2) the rates of mortality* of the "diseased", ascertained both as regards sickness and mortality year by year or at shorter intervals subsequent to the onset of the disease and compared with the corresponding sickness and mortality of the rest of the community.

These preliminary remarks give an indication of the type of work for which we have to look, and we may find examples for consideration in the recent report of the Royal Commission on Venereal Diseases. The report begins with an account of the diseases, syphilis, gonorrhœa and soft chancre, but the last-mentioned is not much discussed in the Report, and needs no direct attention here. Syphilis may be congenital, or may be acquired by direct or indirect contagion; after the primary and secondary lesions, which may disappear even without treatment, a tertiary stage is reached sometimes after the lapse of many years, and the lesions do not then tend to spontaneous healing. Subsequently remote effects follow, and the patient may suffer from nervous affections (locomotor ataxy and general paralysis of the insane), epilepsy, optic atrophy, arterial diseases, &c. The usual treatment was formerly with mercury, but more recently the arseno-benzol compounds (*e.g.*, Ehrlich's Salvarsan) have been used. It is clear that with a disease that may remain apparently quiescent for many years it would be useful to have some test for finding out whether a person has had the disease, and a bio-chemical blood test (Wassermann

* "Rate of mortality" is used in the sense of "probability of death within a year at a particular age", not in the sense of the "death rate of the total population"; the latter is almost useless.

reaction) has proved to be of value in this respect. Syphilis is a frequent cause of antenatal death, miscarriage and still-birth: congenital syphilis* shows itself a few weeks after birth, and the early manifestations have usually run their course in a year if the child survives; later manifestations are peculiarly shaped teeth, forms of eye and ear trouble, meningitis, hydrocephalus, &c.

Gonorrhœa, if not properly treated, may develop into a chronic form, or remain quiescent and subsequently take a gravely injurious active form. It may lead to stricture, various kidney troubles, rheumatism, ophthalmia, &c. It is a frequent cause of sterility.

The descriptions of the diseases and the troubles they carry in their train suggest many interesting statistical inquiries. We want to know how many of the people infected suffer from the various after complaints, whether owing to weakened constitutions they succumb more readily to tuberculosis and other apparently unconnected diseases, what proportion of the infected escape all after-effects, either with or without treatment, and whether, if properly treated, the patient becomes as healthy as if he had never had the disease. We want to know whether it is mere chance that one group of syphilitics develops general paralysis, another tabes, while a third escapes, or do other factors come into the account (*e.g.*, heredity); what proportions of still-births, early deaths, &c., are due to syphilis; and we may also ask how many of the deaths from locomotor ataxy, general paralysis, etc., are due to, or rather follow, syphilis; which treatment is most successful, and whether any treatment leaves permanent ill effects, and we may add "To what extent is the Wassermann reaction a perfect test?"

These problems may sound at first medical rather than statistical: but the solution of each one is only obtained by interpreting the counting of cases, and investigating rates of sickness and mortality. The doctor must decide certain of the preliminary facts, afterwards the problem is statistical.

We may now turn to the statistics considered by the Commissioners and see how far they have been of use in giving the definite information that everyone interested in the subject desires to have. The official statistics furnished by the Registrar-General admittedly cannot give any direct measure of the prevalence of a disease, but an attempt is made to draw conclusions with regard to syphilis from an examination of the deaths due to the diseases that follow it. The deaths recorded in England and Wales from syphilis, general paralysis, locomotor ataxy and aneurysm amount in all to about 5,600 per annum, or 160 per million of the population; but Dr. Stevenson, who presented the Registrar-General's figures, does not regard this as a satisfactory test of the deaths really due to the disease; on the one hand, while we may admit that nearly all the deaths from locomotor ataxy, etc., follow syphilis, there is a residue in which evidence is lacking, and on the other hand some other results of syphilis may be ignored and many deaths in early life

* The Commissioners also make use of the term "hereditary syphilis." This is open to criticism; it is generally held that the syphilis arises from infection before birth.

returned as "congenital debility" are the result of "congenital syphilis." Dr. Stevenson also holds that there is considerable false certification, but this point we can leave for the present, and discuss when we deal with the recommendations of the Committee. We agree with Dr. Stevenson that the statistics are of little use as they stand, and we may go further and say that we fail to see how any useful estimate can be made of the prevalence or increase of such diseases by the study of deaths, although, possibly, we might, after investigating age distribution, be able to say whether the deaths due to the disease have increased. In our view it is not in any way a question of inaccuracy of returns: it is the impossibility of working backwards from the deaths in an attempt to study the statistics of disease A which may have preceded disease B which was the "cause of death." Dr. Stevenson feels apparently that the statistics may be utilised to indicate the distribution of the disease among occupations or districts. We regard the former as useless, on the general grounds that occupation mortality cannot be obtained satisfactorily from the Registrar-General's figures for the reasons with which actuaries are familiar (see *J. I. A.*, vol. xliii, pp. 230-5); the drift of the inefficient, the absurdity of the "clerk" class, and the weeding out of the "sick" from certain occupations, are amongst the many reasons. Apart, however, from such considerations, we are disinclined to think that unsatisfactory data become satisfactory on sub-division. We cannot, therefore, admit Dr. Stevenson's conclusions as regards occupations, and we only agree with his view that syphilis is an urban rather than a rural disease in England because of confirmatory evidence.

We may now turn to the figures produced by the Local Government Board showing the number of still-births, some of which are caused by syphilis, and giving information about the notified cases of ophthalmia due to gonorrhœa. Unfortunately, as regards the former, the notification was uncertain, and the figures in the latter are so erratic that nothing can be done with them. Although we do not think the data can prove anything, it may be mentioned that the smaller towns show a larger proportion of still-births than the larger towns, which is, perhaps, slight evidence against Dr. Stevenson that syphilis is an urban disease, while the other Local Government Board data indicate that gonorrhœa, or rather the ophthalmia often due to it, is more frequent in towns: other possible explanations, however, immediately occur to us.

As regards the community generally, we can get no direct information, but a certain amount of interesting statistical data is furnished by the Navy and Army. In both there is a marked decrease in venereal diseases, although a large number of cases still occurs. The figures could have been made more useful if they had been collected for purposes of investigation instead of as matters of routine, and if they were not weakened by duplication owing to re-admissions. However, there is no doubt that the Navy and Army show a reduction in the amount of these diseases, and this is, at any rate, a cause for some satisfaction.

The Commission also had before it some figures from the Prisons authorities and the Lunacy Commissioners. The inhabitants of prisons can hardly be regarded as a random sample of the population, but the figures are fairly reliable, and give for one prison of able-bodied prisoners capable of performing hard labour 11·67 per-cent showing signs of having suffered from acquired, and 2·16 per-cent from congenital syphilis; in an invalid prison over 24 per-cent, and in a prison containing those sentenced for a serious crime, but not of criminal antecedents, 4·4 per-cent exhibited signs of having had syphilis. In Borstal institutions over 16 per-cent showed signs of congenital syphilis. The figures given by the Lunacy Commissioners were useful in indicating that 10·3 per-cent of males and 1·6 per-cent of females had suffered from acquired syphilis, and ·4 per-cent (sexes combined) from congenital syphilis. It is also stated that syphilis is regarded as a principal cause in 76 per-cent of general paralysis of the insane, 45 per-cent of brain lesions, 43 per-cent of secondary dementia, 34 per-cent of melancholia, 31 per-cent of mania, and 21 per-cent of insanity with epilepsy; but many antecedents considered responsible for insanity depend on syphilis. Incidentally, the statistics afford some evidence in favour of syphilis being more an urban than a rural disease. Of the official statistics produced to the Commission those of the Lunacy Commissioners seem the best; they show an appreciation of problems requiring study, and an attempt to tabulate figures usefully.

The statistical evidence before the Commission included statements from unofficial sources to which reference may be made. A report is published which was received from Dr. Fildes, and related to certain investigations made in the Bacteriological Laboratory (Prof. W. Bulloch) at London Hospital. The object was to ascertain the prevalence of syphilis among persons of the class served by London Hospital and the method used was to invite people to supply a drop of blood "for research purposes": there were few refusals, and about 1,000 cases were examined: 84 gave a positive result when tested by Wassermann's reaction. The conclusion to be based on the result is that from 8 to 12 per-cent of the adult males and 3 to 7 per-cent of the adult females have syphilis. In considering this result is should be remembered that there are relatively few cases, and they relate to a poor and crowded part of London, containing a large proportion of aliens. The statistics attempt, however, to attack a real problem by a direct method, and are of interest and some value. In comparing the various sets of results it must be remembered that there are several variations of the Wassermann test, and the East London figures are obtained by a different test from the others.

Another private collection of statistics was produced by Sir John Collie, and related to (I) 1,119 men on whom a report was required owing to accident or illness; (II) 557 men examined before being employed, and (III) 500 of the same class as II, but submitted to the Wassermann test. In I and II, where clinical evidence was alone relied on, 3·6 per-cent were infected with venereal disease;

in III 9·2 per-cent were "proved to have had syphilis." The class is described as "working-class people over 21 years of age," of a "somewhat superior artisan class," and as many as one-quarter of the 500 had been in the Navy or Army. It seems, from the figures given that if the Wassermann reaction proves syphilis beyond doubt, then the clinical methods considerably underestimate the amount of syphilis; but it is a little embarrassing to find that of 13 persons who admitted syphilis to Sir John Collie, only 4 gave the Wassermann test positive.

Another investigation relates to Wassermann tests carried out in respect of 545 patients newly admitted to 14 asylums in the three months ended 31 December 1914. The asylums were chosen as representing different sections of the population, and it was found that those patients in wholly rural unions showed a lower proportion of positive reactions than those from urban unions. The following table is of interest:

—	Male	Female	Total
Patients tested	275	270	545
Positive and partial reactions	59	34	93
Per-cent of tested	21·3	12·6	17·1
Cases diagnosed as G.P.I.	32	8	40
Positive and partial reactions among G.P.I.	29	8	37

Dr. Mott gave various series of tests, which may be summarized as follows:

Class submitted to Wassermann Test	Per-cent of Positive Reaction		
	Male	Female	Total
Insane (non-G.P.I.), Cane Hill Asylum ...	10·2	...	10·2
Admissions to 3 Poor Law Infirmaries ...	20·9	18·6	19·9

He also gave figures showing that a positive Wassermann reaction is a constant feature in general paralysis.

Statistics were produced by various witnesses showing an undue proportion of miscarriages, infant deaths, etc., among syphilitic families, and Mr. Bishop Harman gave figures showing that instead of 80 per-cent of the children being healthy, as is the case in healthy poor families in London, we have 40 per-cent in syphilitic families. He also produced statistics showing the extent of blindness caused by venereal disease.

Appendix XXII gives the results of the treatment of venereal diseases in the Navy, and deals usefully with the effect of treatment on the Wassermann reactions; some foreign statistics of varying merit are also given.

No reference is made in the report* to the statistics of insurance companies giving some information about the rates of mortality of syphilitics after treatment, a subject unfortunately ignored by the Commissioners; its consideration might have indicated some of the many problems in which statistical work may be utilized.†

We may summarize the statistical work as a whole as unsatisfactory and incomplete—to some extent, unavoidably so: where the statistical treatment is best it refers to special parts of the community, such as the Navy, East End population, lunatic asylums, &c., and we are, therefore, left without knowledge of the facts about syphilis in the general population, and we are still more ignorant in this respect about the other venereal diseases. But in spite of lack of evidence the Commissioners state that "the number of persons who have been infected with syphilis, acquired or congenital, cannot fall below 10 per-cent of the whole population in the large cities, and the percentage affected with gonorrhœa must greatly exceed this proportion." We have searched the report in the hope of finding how this 10 per-cent is reached, but have concluded that it is an "educated guess"; if correct it would seem that syphilitics avoid life assurance, or seldom disclose the disease. Does it also mean that the para-syphilitic diseases less frequently follow the disease than one is led to suppose by the report and evidence?

The first recommendation of the Commission is that there should be confidential registration of causes of deaths. This view was strongly held by Dr. Stevenson (Registrar-General's Office) and by many medical witnesses, but Dr. Dunlop (Office of Registrar-General of Scotland) did not share it and felt that it was sufficient to rely on the growing accuracy of registration, while Sir William Thompson (Registrar-General of Ireland) displayed no enthusiasm for the change.

The arguments in favour of confidential certification of causes of death are that the present method has not led to accurate results, that confidential certification is adopted in nearly all the important continental countries, and that the medical profession would prefer confidential certification as they do not care to let the relatives of deceased patients know the causes of death in certain circumstances. Confidential certification will not, of course, be of any assistance in cases where the doctor makes a mistake in diagnosis, and it is therefore only when he wilfully conceals the true cause of death that private certification can be of assistance. We are inclined to think that the cases of untrue certification are

* Six lines in an Appendix give a reference to the records of the Gotha Company and "other insurance records."

† A suggestion has been made that the exclusion by Friendly Societies of liability in respect of sickness caused by an insured person's own fault deters the person from obtaining efficient treatment of venereal disease. There seems no definite evidence of this. If the basis for premiums excludes part of the sickness from venereal disease then the inclusion of that part might be financially unsound unless the theory that its inclusion will indirectly decrease subsequent sickness is correct.

exaggerated. We do not see why a doctor should object to give a certificate of death from chronic alcoholism, or insanity, etc., if the members of his patient's family know (as they probably do) the cause of death, and in many unfortunate cases a doctor can comfort himself by the reflection that in interpreting a cause of death he naturally reads more significance into the term used than a member of the general public.

The extent of inaccurate certification was discussed by Dr. Stevenson, whose evidence seems to have weighed with the Commissioners, and Appendix I, prepared by Dr. Stevenson, includes a list of specimens of "replies by certifying medical practitioners to enquiries respecting indefinitely certified causes of death."

We may, for the present purpose, leave out the first three of these examples, merely remarking that they are very unconvincing, but we print the last three as they appear in the report.

"4. Female, 32 years.—Certified cause of death 'hæmoptysis.'

"*Reply*.—This patient had suffered from pulmonary tuberculosis "for some time, but owing to her being insured, and as a proviso "against consumption inserted in small print in some forms vitiates "the policy, it is not wise to put in the certificate the fact of "tuberculosis being the cause of death.

"I put on a certificate this year that a lady had suffered from "Bright's disease for 20 years and a fibroid uterus for 12 years, and "it was held to void the policy after paying premiums for six years "in all good faith by a son.

"5. Male, 55 years.—Certified cause of death 'stricture of "œsophagus.'

"*Reply*.—The cause of death in this case was cancer of "œsophagus, but it was not specified because the deceased had "been recently insured, and the insurance company would have "declined to pay up.

"6. Female, 67 years.—Certified cause of death 'pyloric "obstruction.'

"*Reply*.—I believe the disease was carcinomatous, but if I can "help it I do not state in a certificate 'malignant disease', as it "militates against members of the family who wish to insure "their lives."

We cannot understand how anyone reading the "specimens" can find a case for confidential certification. These three examples are examples of poorly-informed medical men who are prepared to certify untruly in the hope that they may enable relatives of a deceased patient to obtain policies of insurance which the medical practitioner thinks it would otherwise be impossible for them to obtain, or to obtain insurance money to which he believes they are not strictly entitled. If the replies are really specimens of the practice of the medical profession in certifying causes of death we can only say that it is a dishonest profession. We believe, however, that there are very few members of that profession who would give a wilfully wrong certificate for the reasons indicated. Insurance companies have considerable experience of individual certificates, and have apparently greater

trust in the honesty and good sense of the medical profession than the member of it who is responsible for printing these very unkind suggestions. There is a touch of humour in two matters connected with these "specimens"—the statement of the certifying doctor in No. 6 as to the practice of insurance companies is so inaccurate that his objection to true certification of the cause of death was groundless; the other touch of humour lies in the suggestion that is made that the necessary alterations in industrial insurance office practice could be overcome by providing out of a first premium of a few pence the fee for a medical examination by one of the members of the profession which according to Dr. Stevenson's "specimens" is helping the general public to swindle insurance companies!

The second argument in favour of private certification seems merely to be based on the idea that it is done elsewhere. We think that if the Commissioners had taken more evidence from insurance officials they would have obtained information about private certification in foreign countries which would have made them less willing to recommend it in the United Kingdom.

The obvious objections of the insurance companies to private certification are that it would mean the alteration of life assurance practice in England with consequent trouble and probable loss, and the opening of the door to fraud, and would cause inconvenience to the relatives of policy-holders. These objections were set clearly before the Commissioners by Mr. Geoffrey Marks, to whose evidence a reference is made in the report in the following terms:

"With a view to ascertain the opinion of the life assurance "companies, whose methods might be affected by such a change, "we examined the manager of the National Mutual Life Assurance "Office, who is also chairman of the Life Offices' Association. This "witness informed us that the question of confidential certification "had not been discussed by the Association, but that his view and "that of two or three other offices were opposed to the change.

"We are not convinced by the alleged objections, and we "consider that any improvement in the accuracy of vital statistics "must, in the long run, benefit the life assurance companies, who "could adapt their methods to the system of confidential certification "as in all other great European countries."

In view of the fact that the Commissioners did not take the trouble to examine any other insurance witness, and as the offices mentioned were all the offices with which the chairman of the Life Offices' Association had been able to communicate, the remark at the end of the first paragraph is a little disingenuous. As regards the second paragraph we must say that it could only have been written by people who had absolutely no information as to the methods by which insurance statistics are formed and handled. A life assurance company seldom requires causes of death in the mass. We do not suppose that the insurance companies would be affected in the slightest degree if massed statistics of causes of death, however accurate, were never again published. What they require is to be able to trace what happens to their individual

policy-holders from the time they are insured until their policies go off the books, and if they require causes of death for investigation purposes it would be to study the causes of death of the individual policy-holders, and this study would have to proceed from the individual cases and not from the Registrar-General's statistics. It is, of course, natural that persons not engaged in life assurance should have little knowledge of its technique, but they might at least refrain from making suggestions which are in contradiction to the only evidence they have taken the trouble to obtain. The real remedy for wilfully inaccurate certification lies in providing penalties for such misdoing. The medical profession is trusted, and deserves to be trusted, in this country with the power of life and death over the community, and we have no doubt that it can be trusted to do all that is required of it, but if there are any unscrupulous members of it, and Dr. Stevenson seems to think there are plenty, the remedy is to be found in punishment, not in encouragement.

The only criticism that we need add to these remarks on the first recommendation of the Commission is that we can find no statement in the report as to the use to be made of the statistics of deaths in the study of venereal disease when perfect accuracy is obtained. We have dealt with this matter at some length because it affects a business with which actuaries are closely connected, but it is only a very small part of the report, and although the recommendation is the first of those mentioned in the Commissioners' summary, this is only because the evidence and remarks leading up to it happen to have been discussed in an early paragraph of the report.

The second recommendation has already been made operative to a large extent, and the next three relate largely to statistical information. Here again there is no indication of the use to which the material is to be put when it has been tabulated, and we feel very strongly that no useful purpose is served by merely obtaining statistics without appreciating for what purpose they are to be used or what information it is desired to obtain from them. It is waste of time and waste of money to produce figures which are "full of sound and fury, signifying nothing." In this connection let us put down one definite criticism. In recommendation 4, it is stated that statistics should be kept of the number of patients for whom salvarsan is provided at the public expense. Of what use is this mere number when it is obtained? If touch is kept with the people who are treated so that the diseases from which they subsequently suffer and the dates of their deaths can be recorded, we might obtain some information with regard to the disease and its treatment, but the mere number of patients is only more troublesome and not more useful than recording the expense to which the State was put for the provision of the remedy. Might not something have been done by the Commission to obtain help from insurance companies or friendly societies so that they could have recommended a method by which people treated would be observed for statistical purposes for a period of years afterwards? It is not an impossible suggestion, but it would require a far larger outlook than the rather useless statistical work which is indicated in the recommendations.

The next recommendations of the Commission are that extended facilities should be available for diagnosis and treatment, and that the cost should be met to a large extent from the Imperial funds and the balance from local rates. We entirely agree that dangerous infectious diseases ought to be treated thoroughly and that treatment should be possible for all, but we do not think public funds should be spent unless the result of the expenditure is watched and its value estimated—this cannot be done by the statistical recommendations of the Commissioners.

Recommendations are made with regard to the treatment of prisoners, Poor-Law patients and men in the Navy and Army, &c.; but, although we see little objection to this, we feel they are minor points in the problem.

The Commissioners seem to have spent a great deal of time in considering also the general education of the public, and if this could be done satisfactorily an improvement in public health might result. We agree wholeheartedly that no man or woman with infectious venereal disease ought to marry, and the education of the public will reduce the evil. There are, however, so many difficulties in the way of education in these matters that the Commissioners have no very definite suggestion to make.

We may sum up our views in the following way: From the point of view of our profession we find that the statistics are open to criticism, and the recommendations with regard to future statistical work are incomplete. With the recommendation of private certification of deaths we have no sympathy, nor do we think that any useful object would be achieved by it. With regard to the attitude of the Commission towards the subject generally, an impression has been left upon our minds that there has been exaggeration. We cannot help the feeling that we have seen a repetition of the well-known case of a Royal Commission in which "The evidence . . . went beyond the facts, the report went "beyond the evidence, the recommendations beyond the report." We hope that the legislation will not go beyond the recommendations, but that if there is any legislation it will revert to the facts, or, if that is impossible in the present state of our knowledge, that it will be framed so that proper provision is made for obtaining facts in such a form as will produce a real estimate of the prevalence and effect of the diseases, and enable us to measure their social importance and the success of various methods of treatment.

W. P. E.

Matemáticas Financieras—Primera Parte—Intereses y anualidades ciertas. By José González Galé.

(Pp. 226. Buenos Aires: Libreria de Antonio Garcia Santos. Moreno y Bolivar. 1916).

During the last two or three years several new books on interest have been published in foreign countries. The second edition of M. Barriol's *Théorie et Pratique* was to a considerable extent new,

and other works have since appeared in Belgium, Italy and now in the Argentine Republic—the latest comer being a Part I to be followed shortly by a Part II on the "Elementos de cálculo actuarial." In all these books will be found differences, not merely of language and arrangement, but also of treatment and application. In Prof. Galé's book the treatment is detailed rather than exhaustive and most of the formulas are illustrated by direct numerical examples. The effect of this seems to us to be to make too much of the formulas, and to obscure the fact that they are themselves merely examples—in symbols instead of numbers—of the application of certain general principles. If formulas without direct examples are not effective in inculcating general principles, we should be inclined to prefer examples without formulas—or at any rate with an irreducible minimum of formulas. Prof. Galé gives a large number of both. We find also in the chapter on Simple Interest (a subject which Prof. Galé follows M. Barriol in taking more seriously than is usual in this country) "rules" for calculating interest for days or months—a 6% method when the year is taken (in accordance with foreign commercial practice) as consisting of 360 days, and another method for the English year, involving the "third tenth and tenth rule" for dividing by 73.*

In the applications to practical finance Prof. Galé deals, among other things, with a kind of loan which is unfamiliar to us—a loan repayable by an accumulative sinking fund, but with an additional lottery element (over and above that involved in the drawings for repayment at par) in the form of bonuses to the first two or three bonds drawn on each occasion. The operation of a loan of this nature is shown by a schedule, and the fact that the bonus is equivalent to an increase in the rate of interest paid by the borrower is clearly brought out. But in the section entitled "Aplicacion del cálculo de las probabilidades a los empréstitos con lotes" Prof. Galé seems to have overlooked the fact that his investigation merely gives the quite reasonable result that the value of the individual bond is the value (per bond) of the entire loan. This can be verified either by transformation of the formula, or by the result of the numerical example, which result will be found to be $1000\, a_{\overline{5}|}^{5\%}/a_{\overline{5}|}^{4\%} + 30\, a_{\overline{5}|}^{5\%}$.

* $\frac{100}{73} = (1 - \cdot 00001)\left(1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}\right)$ nearly.

CORRESPONDENCE.

MORTALITY AMONG NEUTRALS IN WAR-TIME.

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—It has occurred to me that readers of the review, in the April number of the *Journal*, of Dr. Hersch's "La Mortalité chez les neutres en temps de guerre" may be interested to know that the subject was also referred to by Dr. S. Dumas in his paper

"L'Assurance du risque de guerre" published in 1911 in Volume 6 of the "Bulletin de l'Association des Actuaires Suisses."

Dr. Dumas' paper is primarily a paper on war risk, and deals only incidentally with the effect of a war on the mortality of the civil and non-combatant population. For this reason, Dr. Dumas, unlike Dr. Hersch, does not attempt to furnish statistics for different age groups. On the other hand, whereas Dr. Hersch confines his investigations to the "petits pays neutres et limitrophes des pays en guerre, tels que la Suisse, la Belgique et les Pays-Bas" and except in Table IV and diagrams 3–4 deals merely with the number of deaths in each year and not with the ratio of deaths to inhabitants, Dr. Dumas gives the number of deaths (male and female, and male and female combined, except in the case of Germany, where the total deaths only are given) for both combatant and neutral countries, and the number per hundred inhabitants for each year in groups of four years, these groups being taken so as to include the war of 1864—figures for Denmark only—the war of 1866—figures for Germany, Austria, and Hungary—the war of 1870–1—figures for France, Switzerland, Belgium, Holland, England and Wales, and Germany. In a further column Dr. Dumas gives the number of births in each year "pour montrer que l'augmentation de la mortalité correspond souvent à un fléchissement du nombre des naissances ; on sait pourtant que la grande mortalité des nouveaux nés influe d'une manière très sensible sur le nombre total des décès."

In the hope that it may be of interest I reproduce here Table 17, page 93 :

Influence of the Franco-Prussian War on Mortality.

N.B.—The figures for Germany include still-births.

Year	Number of Deaths			Per 100 inhabitants	Number of Births
	Male	Female	Total		
			France.		
1869	442,988	421,332	864,320	2·35	948,526
1870	553,037	493,872	1,046,909	2·84	943,515
1871	692,233	578,777	1,271,010	3·51	826,121
1872	409,811	383,253	793,064	2·20	966,000
			Switzerland.		
1869	34,918	32,231	67,149	2·52	81,766
1870	37,625	35,213	72,838	2·73	83,300
1871	41,866	36,132	77,998	2·92	71,626
1872	33,469	30,273	63,742	2·39	84,313
			Belgium.		
1869	55,768	53,839	109,607	2·21	158,687
1870	60,572	57,787	118,359	2·36	164,572
1871	75,070	70,676	145,746	2·81	158,760
1872	62,041	58,088	120,129	2·35	167,377

Influence of the Franco-Prussian War on Mortality—continued.

N.B.—The figures for Germany include still-births.

Year	Number of Deaths			Per 100 inhabitants	Number of Births
	Male	Female	Total		
		Holland.			
1869	41.751	40,802	82,553	2·29	123,789
1870	47,175	45,891	93.066	2·58	129,997
1871	54,303	52,675	106.978	2·95	128,305
1872	48,550	46,044	94.594	2·59	131,664
		England and Wales.			
1869	254,863	239,965	494,828	2·23	773,381
1870	265,586	249,743	515,329	2·29	792,787
1871	265,563	249,316	514,879	2·26	797,428
1872	255,135	237,130	492,265	2·13	825,907
		Germany.			
1869	...	...	1.154.303	2·85	1,594,187
1870	...	...	1.184.315	2·90	1.634.646
1871	...	...	1.272.313	3·10	1,473,492
1872	...	...	1,260,922	3·06	1,692,227

It will be seen that these figures strikingly confirm the conclusions arrived at by Dr. Hersch.

Dr. Dumas' remarks on this subject are referred to in a volume entitled "Versicherung und Krieg" being a collection of seven papers read at a conference held by the "Deutscher Verein fur Versicherungs-Wissenschaft" on 12 and 13 December 1913, and published by that Society in 1914. In particular Professor Dr. Florschütz showed that Dr. Dumas' figures were confirmed by the experience of the Gotha Life Insurance Company, for whereas the actual mortality was in 1865 only 91·57 per-cent, in 1867 only 90·87 per-cent, and in 1872 only 91·05 per-cent, of the expected, the percentage rose in 1866 to 114·82 per-cent, and in 1871 to 105·10 per-cent. As only 514 policyholders of the Gotha (of whom only 195 kept their policies in force) took part in the war of 1870–71, this striking increase in the ratio of the actual to the "expected" deaths must be put down to the effect of that war on the civil non-combatant population.

Yours faithfully,

D. S. SAVORY.

3, Queen's Gardens,
Ealing, W.,
5 May 1916.

THE SIR GEORGE HARDY MEMORIAL.

To the Editors of the Journal of the Institute of Actuaries.

Dear Sirs,—By an oversight as inexplicable as it is unfortunate, for which I fear I alone am responsible, the name of Mr. A. Digby Besant was omitted from the list given, in the last issue of the *Journal*, of the Committee of the "G. F. Hardy Memorial Fund."

Mr. Besant was one of the small group with whom the idea of the Fund originated. His counsel and assistance throughout the progress of the work, as a member of the Committee, were of the greatest value: and as one of the Honorary Secretaries of the Institute for the year he had a leading share in organizing the memorably successful meeting of January last.

I am afraid this belated acknowledgment can at best make inadequate amends for the omission; but it is all that is possible now, and will at least give the correction as wide a circulation as the mistake.

Yours truly,

S. G. WARNER.

28 *September* 1916.

THE INSTITUTE OF ACTUARIES.

The Sixty-Ninth Annual General Meeting, 5 June 1916.

The President (Mr. Ernest Woods) in the Chair.

The proceedings at the Annual General Meeting will be found on page 159.

REPORT, 1915–1916.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1915–1916, the sixty-eighth year of its existence.

There has been a *decrease* of 28 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-one years later, at 31 March 1906, it was 922. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1907	248	303	383	22	956
1908	253	313	421	22	1,009
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1916.

	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1915	304	361	263	17	945
ii. Withdrawals by					
(1) Death	6	6	2	...	41
(2) Resignation or otherwise	...	12	15	...	
	298	343	246	17	904
iii. Additions to Membership					
(1) By Election	...	...	...	...	13
(2) By Examination	...	...	13	...	
(3) By Re-instatement	...	...	...	...	
	298	343	259	17	917
iv. Transfers					
(1) By Examination:					
from Associates	...	10	...	...	...
to Fellows	10	...	...	...	...
	308	333	259	17	917
(2) By Examination:					
from Students	...	...	12	...	...
to Associates	...	12	...	...	...
v. Number of Members in each class on 31 March 1916	308	345	247	17	917

There are also 172 candidates admitted as Probationers, and 73 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1910 have been as follows:

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1911	160	58	1914	200	67
1912	181	59	1915	188	72
1913	197	55	1916	172	73

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of five Fellows, Messrs. E. J. Bull, R. Cross, R. C. Fippard, A. Smither, and J. Sutherland; four Associates, Messrs. H. C. A. Gravatt, W. M. Haycraft, F. J. Vincent, and D. G. Young; and three Students, Messrs. J. M. Field, A. Jennings, and F. Wellisch. Six of these Members, namely, Captain R. C. Fippard, Lieutenants J. M. Field and D. G. Young, Sergeant F. Wellisch, and Privates H. C. A. Gravatt and A. Jennings were killed in action, and the Council also much regret to have to report that six Probationers of the Institute.

[*Continued on page* 158.]

Dr. *Revenue Account for the*

1915. £	s.	d.		1916. £	s.	d.	£	s.	d.
			Amount of Funds at the beginning of the year—						
9,922	*0*	*3*	General Fund (including Stock of Publications, other than *Journal*)	10,386	14	7			
397	*1*	*9*	Messenger Legacy Fund	409	0	0			
341	*3*	*5*	Brown Prize Fund	351	8	1			
10,660	*5*	*5*					11,147	2	8
...			G. F. Hardy Memorial Fund				767	15	9
			Subscriptions—						
915	*12*	*0*	Fellows	920	17	0			
740	*5*	*0*	Associates	676	4	0			
282	*9*	*0*	Students	247	16	0			
100	*5*	*6*	Probationers	93	19	6			
2,038	*11*	*6*		1,938	16	6			
2	*2*	*0*	*Fines on Reinstatement*	...					
				1,938	16	6			
			Less Waived and returned to Members and Probationers on Naval and Military Service— For the Year 1914–15 . £247 16 0; ,, ,, 1915–16 . 370 13 0	618	9	0			
2,040	*13*	*6*					1,320	7	6
			Entrance Fees—						
8	*8*	*0*	*Associates*	...					
36	*15*	*0*	Students	5	15	6			
25	*14*	*6*	Probationers	13	2	6			
70	*17*	*6*					18	18	0
131	*19*	*6*	Balance of Publications Account				116	19	2
			Dividends and Interest—						
347	*1*	*5*	General Fund	319	19	9			
11	*18*	*3*	Messenger Legacy Fund	12	5	5			
10	*4*	*8*	Brown Prize Fund	10	10	10			
369	*4*	*4*					342	16	0
£13,273	*0*	*3*					£13,713	19	1

Publications Account for the

£	s.	d.		£	s.	d.
252	*16*	*6*	Stock (excluding *Journal*) at the beginning of the year	202	2	5
...			Cost of Revised Edition of Text-Book, Part I	144	10	0
14	*15*	*4*	Binding and Advertising	14	4	3
131	*19*	*6*	Balance	116	19	2
£399	*11*	*4*		£477	15	10

Balance Sheet,

£	s.	d.	LIABILITIES.	£	s.	d.	£	s.	d.	£	s.	d.
10,386	*14*	*7*	General Fund				8,190	4	8			
233	*9*	*2*	Messenger Legacy Fund	233	9	2						
175	*10*	*10*	Accumulated Dividends	187	16	3						
409	*0*	*0*					421	5	5			
200	*0*	*0*	Brown Prize Fund	200	0	0						
151	*8*	*1*	Accumulated Dividends	161	18	11						
351	*8*	*1*					361	18	11			
...			G. F. Hardy Memorial Fund				767	15	9			
										9,741	4	9
80	*17*	*0*	*Examination Fees for year 1915*							...		
12	*10*	*1*	Sundry unpaid Accounts							12	16	8
£11,240	*9*	*9*								£9,754	1	5

year ending 31 *March* 1916. **Cr.**

1915. £	s.	d.		£	s.	d.	1916. £	s.	d.
			Journal—						
561	*6*	*2*	Printing of Nos. 262, 263, 264	369	3	9			
86	*5*	*0*	Editorial Expenses	71	5	0			
647	*11*	*2*		440	8	9			
215	*6*	*4*	*Less* Sales during the year	168	12	6			
432	*4*	*10*					271	16	3
46	*10*	*5*	Library—Binding, Purchases, &c.				31	6	4
59	*5*	*3*	Meetings				36	5	0
329	*2*	*0*	Examination charges	293	14	10			
274	*1*	*0*	*Less* Fees received from Candidates (1915)	135	9	0			
55	*1*	*0*					158	5	10
253	*1*	*0*	Tutors for classes in Parts I and II for 1914-15 and 1915-16	241	10	0			
127	*1*	*0*	*Less Fees received from Students*	..					
126	*0*	*0*					241	10	0
600	*0*	*0*	Office Expenditure—Rent	600	0	0			
416	*7*	*0*	Salaries	449	2	6			
85	*1*	*11*	House expenses	63	19	11			
28	*11*	*2*	Fire and other Insurance	29	0	9			
117	*16*	*9*	Stationery and Printing	89	11	8			
39	*16*	*10*	Postage and Telegrams	33	2	7			
14	*2*	*5*	Sundries	5	8	3			
1,301	*16*	*1*					1,270	5	8
105	*0*	*0*	*Donation to H.R.H. The Prince of Wales' National Relief Fund*				...		
...			Cost of Bust of Sir George Hardy, K.C.B.				130	0	0
...			Amount written off in respect of decrease in value of Stock Exchange Securities				1,833	5	3
11,147	*2*	*8*	Amount of Funds at the end of the year as per Balance Sheet				9,741	4	9
£13,273	*0*	*3*					£13,713	19	1

Examined and found correct, 17 *April* 1916.
ARTHUR TAYLOR,
ROBT. S. B. SAVERY, *Auditors.*
GEORGE H. LAWTON,

year ending 31 *March* 1916.

£	s.	d.		£	s.	d.
197	*8*	*11*	Sales (excluding *Journal*)	180	0	3
202	*2*	*5*	Stock (excluding *Journal*) at the end of the year	297	15	7
£399	*11*	*4*		£477	15	10

Examined and found correct, 17 *April* 1916.
ARTHUR TAYLOR,
ROBT. S. B. SAVERY, *Auditors.*
GEORGE H. LAWTON,

31 *March* 1916.

£	s.	d.	ASSETS.	£	s.	d.
2,370	*0*	*0*	£3,000 Natal 3 per-cent Inscribed Stock	1,875	0	0
1,008	*0*	*0*	£1,200 Metropolitan Railway 3½ per-cent Debenture Stock	874	10	0
1,980	*0*	*0*	£2,000 Great Eastern Railway 4 per-cent Debenture Stock	1,662	10	0
835	*0*	*0*	£1,000 Great Northern Railway Preferred Ordinary Stock	686	5	0
1,431	*0*	*0*	£1,350 Great Western Railway 4¼ per-cent Debenture Stock	1,216	13	9
937	*8*	*0*	£1,000 Dominion of Canada 3½ per-cent Registered 1930-50 Stock	726	5	0
890	*0*	*0*	£1,000 New South Wales 3½ per-cent Inscribed 1930-50 Stock	700	0	0
480	*16*	*0*	£600 Belgian Government 3 per-cent Sterling Loan of 1914	357	15	0
202	*2*	*5*	Stock of Publications (excluding *Journal*) in hand	297	15	7
500	*0*	*0*	*Cash on Deposit Account*	...		
417	*3*	*4*	Cash on Current Account and in hand	527	12	4
189	*0*	*0*	Subscriptions in Arrear	61	19	0
...			£769. 15*s*. 6*d*. War Stock, 4½ per-cent, 1925–45 (G. F. Hardy Fund)	767	15	9
£11,240	*9*	*9*		£9,754	1	5

Examined and found correct, 17 *April* 1916.
ARTHUR TAYLOR,
ROBT. S. B. SAVERY, *Auditors.*
GEORGE H. LAWTON,

Lieutenants C. Bidwell, H. H. Phillips, and G. H. Pollock; Corporal J. B. E. Tombs; and Privates C. J. Elliott and E. H. M. Gumprecht, also died in the service of their King and Country.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,339. 5*s.* 6*d.*, as compared with £2,111. 11*s.* 0*d.* received in the previous year. The difference is mainly due to the fact that the Council have waived subscriptions for the period of the war in the case of Members and Probationers serving with the Forces. The Income and Expenditure for the year were £1,799. 0*s.* 8*d.* and £2,139. 9*s.* 1*d.* respectively.

Three years ago the Stock Exchange Securities held by the Institute were written down to the value of the day. In view of the heavy further depreciation due to the war, the Council have now considered it advisable to write off the sum of £1,833. 5*s.* 3*d.*, and this amount accordingly appears as an item of expenditure in the Revenue Account.

Last year the Council were able to report that over 180 Members and Probationers of the Institute in the United Kingdom and the Colonies had joined His Majesty's Forces. The number has now grown to 300, a figure which includes 39 Fellows and 60 Associates.

Owing to the absence of so many Members of the Institute on war service, and to the great amount of official work which has fallen upon those who remain, the Council decided in November last not to hold any Ordinary General Meetings until further notice. The arrangements that had already been made for the session were therefore cancelled. The Examinations announced to take place this year were also abandoned, and the Classes for Parts I and II discontinued. The usual quarterly issue of the *Journal* has been suspended, but arrangements have been made for the publication of a new number from time to time. The deliberations of the Committee appointed to consider the question of the formulation of a uniform scheme of Pension Fund notation have also been postponed.

The Council have no intention at present of resuming the normal activities of the Institute, but should more favourable circumstances intervene soon enough to enable them to do so, they will at once reconsider the position. An Ordinary General Meeting will, however, be held in November at which the new President will deliver an Address.

A Special Meeting was held on the 11th January, when the G. F. Hardy Memorial Committee handed over to the Council the Fund, amounting to £767. 15*s.* 9*d.*, subscribed to commemorate the life and work of Sir George Hardy. The Council will in due course appoint a Committee to deal with the administration of the Fund. A bronze bust of Sir George Hardy, executed for the Institute by Mr. Gilbert Bayes, was unveiled at the Meeting. A full report of the proceedings has already appeared in the *Journal*.

A revised edition of Part I of the *Text-Book* has been prepared by Mr. R. Todhunter and has been published during the year. Although the changes have not materially altered the general character of the volume, they have involved a large amount of work, and the thanks of the Members are due to the author for the time and labour devoted to the task.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
26,764	Parts of *Journal*.
738	Index to Vols. 1 to 40.
1,873	*Text-Book*, Part I (Revised Edition).
131	*Text-Book*, Part II (Second Edition).

No. of Copies	Description of Work
636	Government Joint-Life Annuity Tables.
738	Select Life Tables.
390	A Short Collection of Actuarial Tables (New Edition).
1,059	Frequency-Curves and Correlation (W. P. Elderton).
39 *in cloth* / 2,331 *in paper*	Lectures on Finance and Law (Clare and Wood Hill).
1,531	Lectures on the Companies Acts (A. C. Clauson).
1,200	Lectures on the Law of Mortgage (W. G. Hayter).
709	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,438	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
931	Lectures on Stock Exchange Investments (J. Burn).
1,513	Lectures on Friendly Society Finance (Sir A. W. Watson).
322	South African War Mortality (F. Schooling and E. A. Rusher).
263	Life Assurance Law (A. R. Barrand).
667	British Offices' Valuation Tables.
665	British Offices' $2\frac{3}{4}$ per-cent Temporary Annuity Values.
643	Transactions of the Second International Congress of Actuaries.
791	Index to Transactions of Seven International Actuarial Congresses.
1,500	Examination Questions, 1911–15.

9 *May* 1916.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Sixty-ninth Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 5 June 1916, Mr. Ernest Woods (the President) in the Chair.

Mr. A. D. Besant (Honorary Secretary), read the notice convening the Meeting and the Minutes of the Sixty-eighth Annual General Meeting, which were confirmed. The Report and Accounts were taken as read.

The President, in moving the adoption of the Report and Accounts, said that when making a similar motion a year ago, he had remarked that no President of the Institute since its foundation had ever had to move the resolution at a more eventful or momentous time. The crisis which then seemed so severe still continued, and its end did not yet appear to be in sight. The battle for right and justice was prolonged, and must be continued until a victorious ending gave them the lasting peace which they and all their Allies were determined to achieve.

He supposed that every one of the members was working to the best of his ability and according to his station in life to assist in the great effort, but a tribute of special honour should be paid to those three hundred men who had gone forth from the Institute to serve their King and country in the Navy and Army. Alas! twelve had already fallen, six members and six probationers, and among them one whose ability and character gave promise

of a brilliant future—Captain Fippard. They had to-day heard of the loss in the North Sea action of Naval Instructor Harry Wallis, who became an Associate in 1913. He was serving in H.M.S. Indefatigable.

They had, too, to regret the loss of other Members who had done good service in the profession. Mr. Smither became a Fellow more than fifty years ago, and, although he had not taken any active part in their proceedings, he was a man of marked ability, and those who had opportunities of meeting him in business could not but recognize his sterling qualities. Mr. Cross died at a comparatively early age. He had served on the Council for eight years, and acted as Examiner on four occasions. His amiable disposition endeared him to all who knew him.

They had recently seen in London Professor Savitch, their Corresponding Member for Russia, who brought good accounts of the efforts of his fellow-countrymen in the common cause, and expressed the regret he felt at the postponement of the Meeting of the Congress in Petrograd. A Corresponding Member for another allied country, France, their old and esteemed friend Monsieur Quiquet, had within the last few weeks written, sending copies of circulars which the Council of the "Institut des Actuaires Français" had been issuing from time to time, giving accounts of their members serving in the French Army, of whom many had distinguished themselves in the face of the enemy, and some had given their lives for their country. He reported that their Belgian Corresponding Members, Messrs. Bégault, Hankar and Lepreux, were in Brussels, and in good health. He (the President) had written twice to M. Bégault, but apparently he had not received the letters.

They congratulated themselves last year on the fact that the Institute had been able to continue its meetings, but as the War and its urgent calls continued, the Council felt that it would be quite out of place to attempt to "carry on" during last session, and as the members were aware they decided that it was their duty to call a halt. That halt was reflected in the accounts, in the slight falling-off in the number of members and in the income, the falling-off in the latter case being mainly due to their having waived the payment of subscriptions for the period of the War in the case of members and probationers serving with H.M. Forces. On the other hand there had been a saving of expense in connection with the *Journal*, and there would be a further saving in connection with the suspension of the Examinations last year, which would appear in next year's accounts.

As to next session, the report contained a promise that the new President would deliver an address in November, and if they elected the gentleman whose name appeared in the ballotting list, he was confident that those who had the good fortune to hear him would enjoy a privilege not to be lightly esteemed. As to the other activities of the Institute, time alone would show whether they could be revived, or whether they would again have to be suspended.

If, however, the public work of the Institute was more or less suspended, private work in difficult circumstances continued, and, in one instance, in a quite unexpected quarter. They had received a communication from the Board of Education asking for assistance in providing books on "Life Insurance", for which the Board had received an application from some British Prisoners of War interned in a camp abroad. All honour to their colleagues who, in the duress of captivity, "endeavour themselves, by way of amends, to be a help and ornament" to their profession.

Although such of their members as were not serving on the active list had had additional labour thrown on them in their official positions, they had not failed in the various tasks outside their ordinary work which had been allotted to them. It would be invidious to mention those who were serving on the various Committees set up by the Government and other public bodies whose names happen to be known, while there were so many others doing just as important work, but work which was not

so prominently in the public view, whose names had not been brought to their notice. He thought, however, that he was justified in referring to the work of Sir Gerald Ryan and Sir Alfred Watson on the Departmental Committee on Approved Society Finance and Administration, the former as Chairman and the latter as Chief Actuary of the National Health Insurance Joint Committee, as he had had the good fortune to be in a better position than others to observe it. The Committee had been sitting for two days a week for three months, and doubtless many of them would have seen the Interim Report recently published. That report was from the nature of the subject very complicated, but he thought that if its recommendations were adopted by Parliament they would go far to remedy the more serious defects which had developed in the practical working of the Act. He hoped that abstracts of the Report, and of the further and final report which the Committee would doubtless make in due course, would be published in the *Journal*.

Mr. W. P. Phelps, in seconding the motion, said the report was essentially a war-time one, and as the President had dealt with it in detail there was no need for him to say much, but he thought that the report might be considered satisfactory. The two chief items of interest were the diminution in income and the diminution in the number of members. The first was very largely due to the result of the action of the Council, which he was sure everyone would approve, and the diminution in members only amounted to 28, and was to be expected. When peace returned he had no doubt the diminution in numbers would no longer exist, and that the membership would show a healthy expansion.

The motion for the adoption of the Report and Statement of Accounts was then put and carried unanimously.

ELECTION OF OFFICERS.

A ballot was then taken for the election of the President, Vice-Presidents, Council and Officers for the ensuing year; and the Scrutineers, Messrs. W. C. Sharman and A. W. Tarn, subsequently reported that the following Fellows recommended by the Council had been unanimously elected:—

President.

Samuel George Warner.

Vice-Presidents.

Lewis Frederick Hovil.
Robert Ruthven Tilt.
Ralph Todhunter, M.A.
Arthur Digby Besant, B.A.

Council.

Thomas Gans Ackland.
Henry James Baker.
Arthur Digby Besant, B.A.
Joseph Burn.
Frederick Timothy Mason Byers.
Charles Ronald Vawdrey Coutts.
William Palin Elderton.
*Oswald Toynbee Falk, B.A.
Duncan Cumming Fraser, M.A.
Lewis Frederick Hovil.
*Charles William Kenchington.
Owen Kentish.
George King.
Geoffrey Marks.
*George Ernest May.
Alfred Moorhouse.
Harry Ethelston Nightingale.
Sir Gerald Hemmington Ryan.
Richard George Salmon.
John Spencer.
Alfred Charles Thorn.
Robert Ruthven Tilt.
Ralph Todhunter, M.A.
Samuel George Warner.
Sir Alfred William Watson.
James Douglas Watson.
*Arthur Thomas Winter.
Ernest Woods.
*William Arthur Workman.
Frank Bertrand Wyatt.

* New Members of the Council.

Treasurer.
Sir Alfred William Watson.

Honorary Secretaries.
Joseph Burn. | James Douglas Watson.

Mr. S. G. Warner (President-Elect), in returning thanks on his own behalf and on that of his colleagues included in the election, said that no event in the whole course of his life had given him greater pleasure than the mark of confidence shown by his election as President. He wished that he were better able to justify it, but to the utmost extent of his power he promised to try. He had personal recollection of those who had filled the office for the last thirty years, and, recalling them to mind, varying widely as they had done in temperament, training and personality, and yet each one devoting to the work his utmost capacity, enriched by the resources of his experience, it was impossible not to feel a strong sense of inheriting a great and worthy tradition. Members met for the present election in unique conditions. Two years ago, when Mr. Woods took office, the catastrophe of war was very near, but entirely unsuspected. In the course of the following few weeks the blow fell, and his (Mr. Woods's) term of tenure had been an anxious and trying one, marked, as all would agree, by a tact and discretion which had been of the highest value. The clouds were still over them, and they were in the midst of their time of trial. Great questions might arise for settlement, on some aspects of which their advice might be sought, and it was in those circumstances a great comfort to him personally to know that in those whom the members elected along with him he had a band of trusted friends, on whose help he could always rely. Speaking on their behalf as well as his own, they felt—and very gladly felt—that they had also in the whole of the members of the Institute a band of friends, common workers with them in the cause of maintaining the high repute of the profession. They entered on the present Institute year a scattered company. Many of their number, as they were proud to know, were standing in the breach against the enemies of the Empire, and risking their lives in its defence. Those who remained at home had their share of responsibility in carrying on the country's work; and that the actuaries' share in that work might grow in influence, in importance and in dignity was the end towards which the efforts of all those whom the members had elected that day would be sincerely directed.

On the motion of Mr. Bernard Woods, seconded by Mr. B. O. Worth, Messrs. Robert S. B. Savery and George H. Lawton were re-elected, and Mr. Wallace Mouat Jones elected, Auditors for the ensuing year.

Mr. Frederick Schooling (Past-President), in proposing a vote of thanks to the President, Vice-Presidents, Council and Officers, including the Assistant-Secretary, for their services during the past year, said that although perhaps not so much ordinary work had devolved upon those gentlemen as would have been the case if the war were not prevailing, nevertheless, as the President had said, they had in their individual capacities much work thrown upon their shoulders. Many of them, including the President, were engaged in most important duties, on Committees and in other ways, and had rendered valuable assistance. Mr. Phelps had remarked that the report was satisfactory. Personally he thought it might be said to be very satisfactory, considering the unique circumstances which the Institute had passed through in common with everyone else. He was extremely sorry the war had prevented the International Congress from being held at Petrograd as arranged, because the President, Mr. Woods, had put in a tremendous amount of hard work in connection with the International Congresses and had represented this country on many previous occasions at those gatherings, and it would have been an appropriate culmination of his

efforts if he could have attended one of the Congresses as President of the Institute. The Secretaries had no doubt had rather an easy time, although they had had to decide some knotty questions, and the Assistant Secretary had, as usual, given his very best aid.

Mr. HENRY COCKBURN (Past-President) in seconding the motion, said he was quite sure that the President and Council in times of difficulty and uncertainty had acted on behalf of the members with all care and circumspection.

The resolution was put to the meeting by Mr. SCHOOLING and carried with acclamation.

The PRESIDENT thanked the members, on behalf of the Council, the Officers and himself, for the kind vote which they had passed. It was perhaps rather ungrateful on his part to take exception to what Mr. Schooling had said in regard to the Secretaries having had rather an easy time during the past year, but, as a matter of fact, that was not the case. There had always been a great deal of work for them to do. He particularly wished to refer to the help the Council always received from Mr. Jarvis, the Assistant Secretary. Mr. Jarvis's clerk had been called up for national service and Mr. Jarvis was doing not only his own work but that of his clerk as well. There was another class of members who were not Officers, who deserved a vote of thanks, namely, those who during the past session had either already written papers, or had gone a long way in that direction, but who did not have the opportunity of reading them before the Institute.

Mr. A. W. TARN, in moving a hearty vote of thanks to the Auditors (Messrs. Taylor, Savery and Lawton) for their services during the past year, said the task had been allotted to him on previous occasions of proposing a similar resolution, but he proposed it on the present occasion with special pleasure, because the senior auditor was one of his colleagues. In ordinary years the Institute had shown its high appreciation of the services of the auditors, who acted in a purely honorary capacity; but in the past year, during such a strenuous time and when the staffs of the offices were so much depleted and so much extra work thus fell upon those who remained, it was a real sacrifice on the part of the auditors to give their time in auditing the accounts of the Institute.

Mr. J. C. PETER seconded the motion, which was carried unanimously.

Mr. GEORGE H. LAWTON, in the absence of his senior colleagues, thanked the members for their kindness in passing the resolution.

The PRESIDENT announced that that concluded the business of the meeting, which stood adjourned to a date thereafter to be settled by the Council.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1915:

	By whom presented (when not purchased).
Accountants and Auditors, Society of Incorporated.	
List of Members, &c., 1915–16.	*The Society.*
Accountants, Institute of Chartered, in England and Wales.	
List of Members, 1916.	*The Institute.*
Acerboni (A.).	
Ensays de una Tabla de Mortalidad de la Poblacion de Buenos Aires. 1916.	*The Author.*

By whom presented (when not purchased).

Actuarial Society of America.

Transactions, 1915–16. — *The Society.*

Containing *inter alia*—

"Premium Loadings and Expense Limitations", by E. E. Rhodes.

"Annuities with participation based upon select and ultimate McClintock Tables", by D. P. Fackler.

"Requirements as to health under applications for reinstatement of Policies. Mortality under reinstated Policies", by A. Hunter.

"Military Service and its bearing on the Policy contract. The recent European War experience of the Mutual Life Insurance Co. of New York", by J. S. Thompson.

"Note on the Mortality Experience of the Mutual Life Insurance Co. of New York", by W. A. Hutcheson.

"A practical rating for Overweights", by A. A. Welch.

"Valuation of Policies grouped as to age attained", by A. D. Watson.

"Mortality Experience of the Worcester Fire Society", by C. R. Fitzgerald.

"Note on Graduation by Adjusted Average", by R. Henderson.

"The European War Risk, with particular reference to the practice and experience of Canadian Companies", by A. B. Wood.

Actuarial Society of New South Wales.

Proceedings, 1916. — *The Society.*

"National Economy", by B. Latham.

"The Valuation of a special type of Benefit Fund", by S. Bennett.

Actuarial Society of Sweden.

Transactions, 1915–16. — *The Society.*

Actuaries, Faculty of.

Transactions, 1915–16. — *The Faculty.*

Containing *inter alia*—

"Inaugural Address", by G. M. Low.

"An analysis of the profit from Endowment Assurance and Whole-Life Policies", by W. A. Robertson and A. G. R. Brown.

Actuaries, Institute of.

Examination Papers: Part I, 1871, 1872, and 1874. Part II, 1871 and 1875. Part III, 1855, 1857, 1867, and 1874–1876. — *A. D. Besant.*

Syllabus of Examinations for the Certificate of Competency, 1873.

American Mathematical Society.

Transactions, 1915–16. — *The Society.*

American Statistical Association.

Transactions, 1915–16. — *The Association.*

Australian Mutual Provident Society.

Sixty-seventh annual report, 1916. — *The Society.*

By whom presented (when not purchased).

Besso (M.).

The "Philobiblon" of Richard De Bury, Bishop of Durham. La. 4to. Rome. 1915. — *The Author.*

"Biometrika."

Volume XI, Parts I, II, and III. — *Purchased.*
Containing *inter alia*—
"On criteria for the existence of differential death-rates", by Prof. Karl Pearson.
"On the correlation between the 'corrected' Cancer and Diabetes death-rates", by C. A. Claremont.
"On the application of 'Goodness of Fit' tables to test regression curves and theoretical curves used to describe observational or experimental data", by Prof. Karl Pearson.

Boag (H.).

Human capital and the cost of the War. 1916. — *The Author.*

Bowley (A. L.).

An elementary Manual of Statistics. 2nd edition. 8vo. 1915. — *The Publishers.*

Braun (Dr. H.).

Extra-Premien voor verblijf van verzekerden buiten Europa. 8vo. Rotterdam. 1915. — *Nationale Levens-verzekering-Bank.*

Brend (Dr. W. A.).

An enquiry into the statistics of deaths from violence and unnatural causes in the United Kingdom. 8vo. 1915. — *The Publishers.*

Casualty Actuarial and Statistical Society of America.

Proceedings, 1915–16. — *The Society.*

Cathles (L. M.).

Is a new Mortality Table necessary? Dallas, Texas. 1915. — *The Author.*

Chappell (E.).

Five-figure mathematical tables, consisting of logs and cologs, illogs, lologs and illologs, together with an explanatory introduction and numerous examples. Also trigonometrical functions with subsidiary tables. 8vo. 1915. — *Purchased.*

Chartered Insurance Institute, Journal of the

Vol. 18. 8vo. 1915. — *The Institute.*
Containing *inter alia*—
"Life saving as a function of Life Insurance", by Dr. E. L. Fisk.
"Reversions and Life Interests", by H. R. Sturt.
"The trend of modern Life Assurance", by W. A. Robertson.
"The Workmen's Compensation Act, 1916: Its amendment and relation to National Insurance", by T. F. Lister.

Collie (Sir John), M.D.

Neurasthenia, what it costs the State. 1916.
The effects of recent legislation upon sickness and accident claims. 1916. — *The Author.*

	By whom presented (when not purchased).
Denmark. Beretning fra Forsikringsraadet for aaret 1914.	*The Danish Government.*
Economic Society (Royal). Journal of the, 1915–16.	*Purchased.*
Eugenics Laboratory Memoirs. XII. Treasury of Human Inheritance, Parts V and VI. 4to. 1911.	*Purchased.*
Fisher (A.). The mathematical theory of Probabilities and its application to frequency-curves and statistical methods. 8vo. 1915.	*Purchased.*
Fisk (Dr. E. L.) Increasing mortality in the United States. New York. 1916.	*The Author.*
Flygare (N.). Meddelanden angäende Försäkringsväsendet i Sverige for ar 1914. 4to. Stockholm. 1916.	*The Author.*
France. Recenil de Documents relatifs aux Assurances sur la vie. No. 2. Réglementation du Contrôle. 8vo. Paris. 1915.	*The French Government.*
Galé (J. G.). Matemáticas Financieras. Primera Parte. Intereses y Anualidades Ciertas. 8vo. Buenos Aires. 1916.	*The Author.*
Gibb (D.). A course in interpolation and numerical integration. 8vo. 1915.	*Purchased.*
Glaisher (J. W. L.). Report of Committee on Mathematical Tables. 8vo. 1873.	*J. Spencer.*
Gumbell (E. J.). Die Berechnung des Bevölkerungsstandes durch Interpolation. Leipzig. 1916. Die Versuche eines mathematischen Gesetzes der Bevölkerungszunahme. München. 1916.	*The Author.*
Henderson (R.). Mortality Laws and Statistics. 8vo. 1915.	*Purchased.*
Hendry (W.). The method of calculating the values of Life Annuities, Assurances, &c., with Tables. 8vo. Hull. 1820.	*H. Cockburn.*
Hepburn (A. B.). History of Coinage and Currency in the United States, and the perennial contest for sound money. 8vo. New York. 1903.	*London School of Economics and Political Science.*
Hersch (Dr. L.). La Mortalité chez les neutres en temps de guerre. Paris. 1915.	*Purchased.*

By whom presented (when not purchased).

Hoffmann (Dr. F. L.).

The mortality from Cancer throughout the world. 8vo. Newark, N.J. 1915. — *Prudential Insurance Co. of America.*

Holland.

Archief voor de Verzekerings-Wetenschap. 1915–16. — *The Society.*

Mededeelingen der Vereeniging voor Levensverzekering. — *The Society.*

Dordrecht Levensverzekering-Maatschappij. Verslag over het jaar 1915. — *The Company.*

Nationale Levensverzekering-Bank. Verslag over het jaar 1915. — *The Company.*

Institute of Actuaries Students' Society.

Journal of the. Vol. I. No. 5. 8vo. 1915. — *The Society.*

Institute of Bankers.

Journal of the, 1915–16.
Catalogue of the Library, 4th edit. 8vo. 1915. — *The Institute.*

Insurance Institute of America.

Proceedings of the Seventh Conference. 8vo. 1915. — *The Institute.*

Insurance Institute of London.

Journal of the. Vol. VIII. 1914-15. — *The Institute.*

Insurance Institute of Toronto.

Proceedings, 1915–16. — *The Institute.*
Containing *inter alia*—
"Lapsed, surrendered, and not-taken Policies", by B. W. N. Grigg.
"International Exchange", by M. A. Mackenzie.

Italy.

Bolletino di Notizie sul Credito e sulla Previdenza. — *The Italian Government.*

Japan.

Japanese Insurance Year-Book, 1913. 8vo. — *A. J. Cook.*

Knibbs (G. K.) and Barford (F. W.).

Studies in statistical representation, III. Curves, their logarithmic homologues and antilogarithmic generatrices. Sydney. 1915. — *The Authors.*

Life Offices' Association.

Minutes, etc., of the Meetings of the Standing Committee, 1915–16. — *The Association.*

London Mathematical Society.

Proceedings, 1915–16. — *The Society.*

Maluquer y Salvador (J.).

Union Internationale des Assurances. Madrid. 1916. — *The Author.*

Mittra (B. B.).

Manual of Life Assurance. 8vo. Calcutta. 1916. — *The Author.*

Mutual Life Insurance Co. of New York.

Exhibit, Paris Exposition, 1900. 4to. New York. 1900. — *The North British and Mercantile Insurance Co.*

	By whom presented (when not purchased).
Napier Tercentenary.	
Memorial Volume, edited by C. G. Knott. 4to. 1915.	*Purchased.*
Olifiers (E.).	
Valuation of the Death Benefits provided by the Workmen's Compensation Law of New York. New York. 1915.	*The Author.*
Oliver (Sir Thomas), M.D.	
Occupations from the social, hygienic and medical points of view. 8vo. Cambridge. 1916.	*Purchased.*
Parliamentary Papers.	
Acts of Parliament.	
Consolidated Fund, 1915. Finance, 1915.	*Purchased.*
Assurance Companies. Returns to the Board of Trade. 1915.	*Purchased.*
Censuses.	
1911. England and Wales. Summary Tables. Fol. 1915.	*The Registrar-General.*
Colonial Acts and Bills.	
Canadian Insurance Act, 1910, with the Winding-up Act and Regulations of the Treasury Board regarding the acceptance of Securities as Deposits from Insurance Companies. 8vo. Ottawa. 1910.	*The Government Insurance Department.*
Life Assurance Companies. Victoria. 1873.	*The North British and Mercantile Insurance Co.*
Colonies.	
Canada.	
Report of the Superintendent of Insurance for the year 1914.	*The Government Insurance Dept.*
New South Wales.	
Friendly Societies. Report of the Registrar for 1914. Official Year Book, 1914–15. Statistical Register for 1914 and previous years. Vital Statistics. Report for 1914 and previous years.	*The Government of N.S.W.*
New Zealand.	
Friendly Societies. Thirty-eighth Annual Report of the Registrar, 1914. National Provident Fund. Actuarial examination for the period ended 31 December 1913. 1916. Official Year Book, 1915. Public Superannuation Fund. Actuarial examination for the triennium ended 31 December 1913. 1915. Statistics of the Dominion for the year 1914. Teachers' Superannuation Fund. Actuarial examination for the triennium ended 31 December 1913. 1915.	*The Government of N.Z.*

	By whom presented (when not purchased).
Parliamentary Papers—(*continued*).	
Victoria.	
Friendly Societies. Thirty-seventh Annual Report of the Government Statist, 1914. Twenty-sixth Annual Report of the Registrar, 1915. Statistical Register for 1914.	*The Government of Victoria.*
Western Australia.	
Friendly Societies. Report of Proceedings by the Registrar for the year ended 30 June 1915.	*The Government of W.A.*
Companies. Twenty-fourth Annual Report by the Board of Trade. Fol. 1915. Friendly Societies. Building Societies. Twentieth Annual Report of the Chief Registrar for 1914. Chief Registrar's Reports for 1914. Government Insurance and Annuities. Account to 31 December 1915.	*Purchased.*
National Debt. Return showing Liabilities and Assets at the close of each Financial year from 1835–36 to 1915–16, both inclusive.	*Purchased.*
Public Trustee. Eighth Annual Report, 1916.	*Purchased.*
Registrar-General. England. Seventy-seventh Annual Report of Births, Deaths, and Marriages, 1914.	*The Registrar-General.*
Registrar-General. Ireland. Fifty-first detailed Annual Report of Births, Deaths, and Marriages, 1914. Registrar-General. Scotland. Sixtieth detailed Annual Report of Births, Deaths, and Marriages. (Abstracts of 1914.) Venereal Diseases. Report of the Royal Commission on, with Minutes of Evidence and Index. Fol. 1914–16. Workmen's Compensation. Statistics of Proceedings under the Acts of 1897 and 1900, and the Employers' Liability Act, 1880, during the year 1914.	*Purchased.*
Pearl (R.).	
Modes of research in Genetics. 8vo. 1915.	*Purchased.*
Pedersen (J.).	
Om et Livsforsikrings-selskabs okonomi. 8vo. Copenhagen. 1915.	*The Author.*
Periodicals.	
Accountants' Magazine. American Bankers' Magazine.	*Purchased.*
American Economic Review.	*The Editor.*
Bankers' Magazine.	*Purchased.*
Bankers' Magazine. 1907.	*H. W. Andras.*
Economist.	*Purchased.*
Insurance Record. Journal of Political Economy. (Chicago). Post Magazine. Post Magazine Almanack.	*The Editor.*

	By whom presented (when not purchased).
Phœnix Assurance Co., Ltd., 1782–1915. Souvenir Volume. Ob. 8vo. 1916.	*The General Manager.*
Porter (G. R.). The Progress of the Nation in its various social and economic relations from the beginning of the Nineteenth Century. New Edition. By F. W. Hirst. 8vo. 1912.	*Purchased.*
Schloss (D. F.). Insurance against unemployment. 8vo. 1909.	*The London School of Economic and Political Science.*
Scientific and Learned Societies of Great Britain and Ireland. Official Year Book, 1915.	*Purchased.*
Sellar (A. S.). Tables for computing Income Tax at 3/6 in the £, showing Income Tax and also net amount at one inspection. 8vo. 1915.	*The Author.*
Stamp (J. C.). British Incomes and Property. The application of official statistics to economic problems. 8vo. 1916.	*Purchased.*
Statistical Society (Royal). Journal of the, 1915–16.	*The Society.*
Stock Exchange Official Intelligence, 1916.	*Purchased.*
Stoltz (G.). Den Statliga Sjukförsäkringen i England. Stockholm. 1915.	*N. Flygare.*
Stuckey (J. J.). Table of Compound Interest at $\frac{1}{8}$ per cent. and of antilogarithms to sixty figures to base 1·00125. 4to. 1915.	*Purchased.*
"Sunshine." Memorial Number, commemorating Robertson Macaulay's Presidency of the Sun Life Assurance Co. of Canada, 1873–1915. Montreal. 8vo. 1915.	*T. B. Macaulay.*
Sweden. Enskilda Försäkringsanstalter ar 1913, 1914, av Kungl. Försäkringsinspektionen.	*The Swedish Government.*
Swedish Insurance Society. Transactions, 1916.	*The Society.*
Switzerland. Rapport du Bureau Fédéral des Assurances sur les Entreprises Privées en Matière d'Assurances en Suisse, 1914. Schweizerische Unfall-Versicherungsanstalt in Luzern. Erläuterungen zu den ersten Prämiensätzen der obligatorischen Betriebs-Unfallversicherung. Prämientarif. Die Elemente zu Berechnung der Renten-Deckungskapitalien. Luzern. 1916.	*The Swiss Government.*

By whom presented (when not purchased).

Ulmgren (P.).

Nagra ord om Abbrottsförsäkring. Stockholm. 1915. — *N. Flygare.*

United States of America.

Official Publications.

The Federal Registration Service of the United States: Its development, problems, and defects. By Dr. C. L. Wilbur. 8vo. Washington. 1916.

Fifteenth Annual Report of the Bureau of the Census on Mortality Statistics, 1915. La. 4to. Washington.

United States Life Tables, 1910. 4to. Washington. 1910. — *The U.S. Government.*

Another Copy. — *H. Moir.*

Connecticut.

Fifty-first Annual Report of the Insurance Commissioner, 1916. — *The Commissioner.*

Massachusetts.

Sixty-first Annual Report of the Insurance Commissioner, 1915. — *The Commissioner.*

New York State.

Fifty-seventh Annual Report of the Superintendent of Insurance, 1916. — *The Superintendent.*

Report on the Teachers' Retirement Fund, City of New York. 8vo. New York. 1915.

Report on the Pension Funds of the City of New York. Part I: Operation of the nine existing Pension Funds. Part II: An actuarial investigation of the Mortality and Service Experience of the Special and General Service Funds for Municipal Employees. 2 vols. 8vo. New York. 1916. — *The Commission on Pensions.*

Weld (L. D.).

Theory of Errors and Least Squares. 8vo. New York. 1916. — *Purchased.*

Withers (H.).

International Finance. 8vo. 1916. — *Purchased.*

Additional copies of works already in the Library:

Journal of the Institute of Actuaries.

Vols. XXIV to XXXII inclusive. — *"An F.I.A."*

Life Assurers' Handbook, 1877, 1878. 8vo. — *North British and Mercantile Insurance Co.*

Parliamentary Papers.

India.

Returns of Life Assurance Companies doing business in British India, 1914. Fol. Simla. 1915. — *G. King.*

Obituary.

Mr. T. G. Ackland.

It is with great regret that we have to record the death, on 28 June 1916, of Mr. Thomas Gans Ackland. Although he had not been in good health during the last three or four years, and had been obliged to resign the editorship of this Journal, he continued his consulting practice, and the active interest which he took in actuarial matters seemed to justify the hope that he would be able to do good work for some years to come. For Mr. Ackland was emphatically a worker, and the Institute has lost in him not only a Fellow of 40 years' standing, a former Vice-President, an ex-Treasurer, and a member who had received the distinction of the Honorary Fellowship of the Faculty, but also one who, throughout his membership, was an earnest student of actuarial science.

Mr. Ackland was born in 1851, and was elected an Associate of the Institute in 1869. The first 25 years of his business life were spent in the service of the Gresham Life Assurance Society, of which he became Actuary in 1883 and Manager in 1888, and it is to his connection with that Society that we are indebted for his first important contribution to actuarial literature—the well-known Report by himself and Dr. Smee on War Risks and Extras. At an earlier period of his official connection with the Gresham, he had acquired some academic reputation by his paper on the Summation Method of applying Woolhouse's Graduation Formula, and had filled for a time—from 1883 to 1886—the position of Part II Lecturer at the Institute—a practical teaching experience which led to his collaborating with the late Sir George Hardy in the compilation of "Exercises and Examples."

The insurance period of his career was not, therefore, by any means unproductive from an actuarial point of view, but the administrative responsibilities of an insurance company were not, perhaps, best suited to his temperament and abilities, and it was during the 20 years after he left the Gresham that his name became widely known, and that he did his best work. In the varied activities of consulting practice he found his vocation. As a consulting actuary to insurance companies he was eminently successful—firm in upholding actuarial principles and in the support he gave to the permanent actuarial staff, yet, at the same time, not insensible to practical managerial difficulties, and with a mind open to conviction when conviction was consistent with what he believed to be sound finance. And the qualities which made him a good consultant were of great service to him in the responsible and by no means easy position of Actuarial Adviser to the Board of Trade. In this capacity he had to deal during recent years with two matters of the first importance, namely, the legislation affecting insurance companies and the Unemployment part of the National Insurance Scheme. His Reports on the latter were published in the *Journal*.

While the occupations just mentioned afforded scope for the exercise of Mr. Ackland's judgment and knowledge of affairs, the new problems of private practice, and still more, perhaps, his appointment in 1896 as Official Supervisor of the British Offices Mortality Investigation, gave a marked impulse to his natural inclination for academic work. Shortly before his appointment as Supervisor of the Mortality Investigation, he had submitted to the Institute a voluminous paper on methods of determining rates of mortality and withdrawal and the application of such methods to the special case of Clerks' Associations, and this paper, in conjunction with the paper which he submitted in 1902 on the British Offices' Tables and with the first part of "Principles and Methods," give a sufficient indication of the immense amount of time, thought and labour he devoted to the questions involved in the Investigation. The two papers we have mentioned, in common with his full dress papers on other subjects, may be felt to be over-elaborated, but they cannot fail to be recognized and admired as good honest work executed with conscientious thoroughness. It was not in Mr. Ackland's nature to pass over a doubtful point without investigating it, and if it was a point that lent itself to detailed algebraical and numerical elucidation he was the more attracted by it. His methods may not always have been elegant, but they were fine examples of courage and perseverance. Moreover, they afforded more than once (in unpublished as well as published work) occasions for the exercise of one of his best qualities—a generous readiness to recognize and welcome improvements upon his own methods. Few men have been less prejudiced in favour of their own ways of doing things. We may recall two instances in connection with the paper on Approximate Valuation with Allowance for Selection—his cordial appreciation of Mr. Lidstone's method of obtaining the coefficients in the formula for the difference between the select and non-select probabilities, and his handsome acknowledgment (as Sir George Hardy characterized it from the Chair) of the superiority in certain respects of the method of valuation subsequently suggested by Mr. E. H. Brown. A closely-related and not less attractive trait was his prepossession in favour of original work, especially if it were of a mathematical type. There was a certain fresh enthusiasm—associated more often with youth than with middle age—in his attitude towards new work. Without any thought of appropriating any part of the credit of it to himself, but with the zeal of a student in furthering actuarial science, and perhaps a certain pride in being "in" the advanced movement, he was as ready to help in developing the ideas of others as most men are to develop their own. Thus he joined actively in the exploitation of Z—applying it to joint life endowment assurances; he frequently used and applied the ideas of the late Sir George Hardy, for whose work he had the warmest admiration; he employed the method of moments for the determination of constants; in one of his last and most important investigations—on the subject of the Indian Census—he adopted the frequency curve method for the distribution

of the populations, and even if in this case his belief in modern methods should be thought to have led him into an error of judgment, it was an error which may be readily forgiven for the sake of the sound instinct which prompted it.

Mr. Ackland was a frequent speaker at Institute Meetings, especially in recent years, and his intervention in a debate was always welcome, not on account of any oratorical gift of style—although he spoke fluently and with an unaffected simplicity which is one of the essentials of style—but because he had almost invariably something useful to contribute to the discussion. His speeches gave the impression, whether on delivery or in the published abstract, that he had done the author the justice of studying his paper, and that he had endeavoured to enter into his point of view. He was a candid, but courteous critic when he believed that criticism was justified, but he started with a presumption in favour of a paper and would devote himself loyally to developing its methods or principles and to suggesting further applications.

One of Mr. Ackland's services to the Institute that deserves special mention is his work in connection with the Examinations. Probably no member of the Institute has examined on so many occasions, and none ever undertook the task more readily or did more willingly his fair share of the labour involved. Whether from the point of view of the candidate or from the not less important point of view of his colleagues, he was an excellent Examiner. His teaching experience and his sympathy with students (of whom he was always at heart one) inclined him to temper justice with mercy in the one capacity, while his openness of mind and willingness to compromise rendered him most acceptable in the other. We are glad to believe also that he found a real pleasure in making up questions, especially on the more theoretical subjects. For Interpolation he had a special affection, and it may be recalled that his first and last contributions to the *Journal* were on this subject.

We may be permitted to add a few words on Mr. Ackland's association with the *Journal.* Among the many activities of his professional life there was probably none to which he attached more importance, or to which he devoted more time and energy, than the Editorship of the *Journal.* During the seven years over which his Editorship extended he personally prepared in almost every case the Abstracts of the Discussions, expending on them very considerable time and thought. But of more importance than the painstaking fidelity with which he performed this routine work was the conscientious labour he bestowed on anything of the nature of an original contribution. Here his genuine interest in actuarial research found scope. He did not, as a rule, rely solely on his own judgment of such contributions. Yet when he wrote submitting some new contribution to a member of the profession whom he considered specially qualified to pronounce upon it, his letter, while deferring with characteristic modesty to the opinion of the person to whom it was addressed, would show invariably that he had given

the contribution careful consideration and often that he had anticipated all that could be said about it. His services as Editor were rendered in a truly scientific spirit, and the expression of appreciation with which the Council of the Institute acknowledged them when announcing his resignation may be recalled as a fitting tribute to his memory.

Walter Sydney Emery, Student of the Institute, Private, Infantry Battalion 1st Australian Expeditionary Force.

Killed in Action 8 *August* 1915.

John Morton Field, Student of the Institute, 2nd Lieutenant, Royal Sussex Regiment.

Killed in Action 11 *April* 1916.

Harry Wallis, M.A., Associate of the Institute, Naval Instructor, H.M.S. "Indefatigable."

Killed in Action 31 *May* 1916.

Donald Kerr, M.A., B.Sc., Probationer of the Institute, 2nd Lieutenant, 2nd Battalion London Scottish Regiment.

Killed in Action — *July*, 1916.

Ernest Charles Kemp, Student of the Institute, Lieutenant, 9th Service Battalion Yorkshire Regiment, and Royal Flying Corps.

Killed in Action 6 *September* 1916.

George Henry Grantham, Student of the Institute, Private, 15th Battalion County of London Regiment.

Killed in Action 15 *September* 1916.

Sidney Frank Snowdon, Student of the Institute, 2nd Lieutenant, 1st Battalion London Regiment, Royal Fusiliers.

Killed in Action 15 *September* 1916.

Leslie Davies, Probationer of the Institute, Captain, 15th Battalion County of London Regiment.

Killed in Action 17 *September* 1916.

SESSIONAL MEETINGS OF THE INSTITUTE.

A Meeting of the Institute will be held on Monday, 27 November, when the President (Mr. S. G. Warner) will deliver his Inaugural Address.

There will be no Meeting in December, but on 29 January 1917 a Meeting will be held to hear and discuss a Lecture by Mr. Hartley Withers on some financial topic of current interest. The precise subject of the Lecture will be announced later.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Opening Address by the President, S. G. Warner, Esq.

[Delivered 27 November 1916.]

Gentlemen,—In addressing you to-night my first duty is to thank you for the honour conferred upon me by my election to this Chair, which so many illustrious members of our profession have so worthily filled; and to express the hope that I may be able so to discharge the responsibilities its occupancy involves as to maintain unsullied its high tradition.

In accordance with appropriate and gracious custom, we have first to pay tribute to the memory of those who, since last we met together in this Hall, are with us no longer. In Mr. T. G. Ackland we have lost one whose work for actuarial science had been untiring and persistent throughout a generation. In speaking of the late Mr. Manly, my predecessor, Mr. Woods, referred to him as one who might be said to have lived for the Institute. The same phrase seems appropriate in the present case. As Lecturer, Examiner, Member of Council, Vice-President, Treasurer, Editor of the *Journal*, Honorary Supervisor of the last great mortality investigation undertaken by the Institute, the name of Mr. Ackland is honourably remembered by us all. In addition to this we recall his valuable services to the Government of India in connection with its census work, and to our own Government in the construction of the Unemployment Insurance scheme: also as actuarial adviser to the Board of Trade. Throughout this career of untiring professional labour his high character, geniality, and ready friendship for all professional colleagues, young or old, made it a pleasure to

know him. We mourn his loss as that of one who will not be easily replaced.

Mr. F. W. Frankland was with us as an actuary working in England for but a short time; and it is now a good many years since he left our shores to pursue his professional career elsewhere. Some of us, however, have pleasant and friendly memories of him during his stay in this country. Most of his actuarial work was done in New Zealand and in America. He was a man of powerful and varied mental endowment, with an interest in philosophical as well as actuarial questions, and his death deprives our profession of one who in many ways adorned it.

In Mr. Emory McClintock, the famous American actuary, we have one whose name and work are so widely known as to need little notice here. For upwards of forty years he had been a Fellow of our Institute, and may thus be ranked among the veterans of the profession. His friendship for British actuaries was sincere and constant, and we join with our professional brethren beyond the Atlantic in regretting his loss.

Within the last few weeks we have lost by death Mr. Joseph Stocks, one of the ablest and most active of our younger Fellows. As it happens, I was for some years personally associated with him in business, and can bear witness to his brilliance of mind and enthusiasm of temperament. He had much personal charm, and overflowing energy; and his work as Honorary Secretary of the Students' Society, where he was best known, was of very great value. By a wide circle of friends he will be held in affectionate remembrance.

To these names, illustrious because of service rendered, we give due homage. There are others, to whom length of days in the career they had chosen has not been appointed, who have fallen at its threshold, and around whose memory there shines a light of sacrifice. Young, brave and ardent, they have answered the call of duty, have taken arms in their country's cause, and have died in her service.

Since we last met ten more names have thus been added to that list which will always be one of our honoured possessions. This makes a total of 24 (exclusive of four reported as "missing") out of the 335 of our members who are on active service. Such an occasion is not one for the commonplaces of rhetoric. To those whom these young men have left bereaved and sorrowing, we tender our sympathy. With the fallen, it is well. No strife or turmoil can vex them further. They are at peace.

Their example remains to inspire those by whom the struggle must still be carried on.

It is that struggle which must form the keynote of what we have to say to each other to-night. It cannot be otherwise. When my predecessor addressed us from this chair two years ago, the War was upon us, and we had already undergone some of its sharpest experiences. But it had lasted a few months only. The possibilities of its future were but dimly discernible. Now the months have become years;—resources, material and personal, which dwarf all precedents in history, and which the imagination can but feebly grasp, have by all the combatants been poured into the caldron,—and still the War goes on. It is changing the future conditions of existence in ways which we cannot measure yet, and may not appreciate in all their profundity for many years. We are glad to believe that now, thanks to the heroism on land and water of our countrymen, our kinsmen from overseas, and our allies, we are well on the road along which persevering progress will bring us to the only end that will justify the sacrifices made; but the end is not yet.

In these circumstances memory naturally turns back to the one time in our country's history which comes nearest to affording a parallel; and which, as it happens, is separated from us by just over a century. When the United Kingdom emerged in 1815 from the European conflict, which, with little interruption, had lasted for twenty years, it found itself in a state of severe exhaustion, faced with an unprecedentedly large national debt, and with the prospect before it of a long period of peace. The intervening century, owing to the fertility of scientific discovery and its practical application, has altered the conditions of civilised life more profoundly than the ten centuries preceding. Change and development have proceeded accordingly at a staggering pace, and with mixed results. Then has come, with dramatic suddenness, the old ordeal again, but magnified to Titanic proportions. We may look upon it as a putting to the test, in all its aspects, of the results of the century's activity. Of each institution, organization, development, tendency, the crisis asks:—"What are you qualified, able, ready to do for the nation in its evil day? What, if any, contribution of strength or help or guidance can you bring in to the common stock?"

We are concerned here with one of these national assets, that of actuarial science. What is now proposed is, after briefly following its development during the period, to set out as far as possible the kind of service which it has been able to render at the time of national need; to consider some present practical questions in which its interests are involved; and to attempt some forecast as to its place and powers in the work of the future. It seems desirable on such an occasion as the present, when the conditions are in so many ways unique, to introduce as little technicality as possible, and to take, so far as one can, rather a broad and general view of the field. If in doing so some things may be said, and some considerations introduced, which are not strictly speaking actuarial, the defence must be that we are citizens first and actuaries afterwards, and that sometimes an object of observation cannot be rightly understood if isolated from the surroundings amid which it is naturally set.

It is difficult to form any satisfactory mental picture of the contrast between our country as it was in 1815 and as it is to-day; the difference in every direction is so enormous. A few typical figures for England and Wales may afford a little help. In the interval the population has increased fourfold. In 1815 there were 13 public railways completed, of which the longest measured 26 miles.* Local expenditure for public purposes was represented by an annual sum of less than £8,000,000, as compared with £166,000,000 in 1911. There was no national expenditure on education. Such work as was done in this direction, as late as 1833, was entirely that of the voluntary organizations known as the "National" and "British" Schools; and the modest efforts of the Societies supporting these institutions were criticized in influential quarters as "undermining the foundations of all property."

Labour suffered from penal restrictions against leaving the country, and from great discouragement of all moving from place to place; but the outstanding thing which strikes one in this connection is that we were still largely an agricultural people, and one of small towns. The industrial revolution had begun, but it was yet in its childhood. In 1811 the percentage of the population supported by agriculture has been computed as 34 per-cent; in 1911, as 8 per-cent. In 1821, there were in England and Wales only eight towns with more than 50,000

* These were worked chiefly, if not wholly, by horse-power. Steam traction was still in its tentative and experimental stages.

inhabitants, representing 16 per-cent of the population. In 1911, there were 98, representing 48 per-cent.*

The foundations of actuarial science had been laid, and laid soundly. As it happens, it was in the year 1815 that the famous Carlisle Table was published. Notwithstanding all subsequent developments of the principles on which it was based it still holds its own as a useful and practical instrument in the hands of the actuary. In this it may be taken as typical of the work of those pioneers of our profession whose steady and thorough examination of its principles enabled them to make for their successors, through regions hitherto untrodden, a sure and stable road. They had to proceed in many ways independently, and without the enormous advantage of uniform modes of expression. Their notations have become obsolete, and so they speak to us in unfamiliar language, which in these crowded days makes the study of their utterances a pursuit of antiquarian interest for which few of us have leisure; but none the less let them be held in honour as those who made further progress possible.

When we consider what is the great practical achievement which actuarial science has contributed to the development of the country's resources during the century which has elapsed since 1815, there can be but one answer. It is the business of British life assurance, as we know it to-day. Whatever else there may have been, this in magnitude and importance overshadows it. That business was in vigorous existence at the beginning of the period. Several honoured companies which are still with us, as well as others now no more, were conducting it with activity. At the same time, while it is not possible to give figures showing the progress of life assurance in 1815, it is certain that the result, if we had it, would be negligible in comparison with the totals of our last Parliamentary returns, showing sums assured of over 1,300 millions, premium income of 50 millions, and invested funds of about 450 millions sterling.

We are not doing anything incongruous in thus passing from talk about a science to talk about a business. I think, indeed,

* For some of the material here used I should like to acknowledge my indebtedness to a very able essay by Mr. H. R. Hodges on the Economic condition of England in 1815 and 1915, which won the London University Reitlinger prize for 1915. It is at present in manuscript, and the property of the Library of the London School of Economics, by the courtesy of whose Librarian I have had access to it for reference.

that we should accustom ourselves a good deal more to recognize the relation freely and frankly. It is true that we here have the honour and privilege of representing a science, and the thought can never be too fully before us. It is none the less true, however, that many of us are daily occupied in the commercial enterprise which that science has made possible; and we are likely to render the best service to each by maintaining the consciousness of our responsibility to both.

There will, perhaps, be few things more interesting to the future historian of the marvellous development of British industry and finance during the last hundred years than the chapter which will record the growth of life assurance. The period was one of enterprise and promise in all directions. Fresh paths, hitherto undreamed of, for the employment of capital were opened up. In such circumstances the natural desire was to put as much money as possible into the venture, and a plan which presented the possibility, at a moderate cost, of avoiding the disaster to dependants of premature death, left the way clear for a maximum effort.

At the same time there came into existence, and grew rapidly, the great middle class; professional men, employees and others whose sole capital and means of support for their families lay in their own health and energies. To these also the principle of co-operative protection at an average cost against a risk too heavy for individual effort made an irresistible appeal. Never has there been a more striking instance of the bringing together of a want to be met and the means of meeting it. Amid the expansion which by leaps and bounds was transforming the entire aspect of society, the business of life assurance struck its roots deep and wide; vindicating its right to exist and making itself indispensable to the progress of the community.

The more we consider this the more I think we shall be disposed to agree that here we must fix the essential sphere of activity and achievement of the actuaries of the nineteenth century. We are looking back, remember, upon times when the work of an assurance office dominated the career and the outlook of entrants into our profession more exclusively than it does to-day. The two fields of action were far more largely co-terminous. The task that lay before the actuaries was no light one, and their record of accomplishment is one of which we need not be ashamed. What a great deal had to be done, we are apt to forget, until we review it in that record. A science which

had no agreed authoritative basis, even as concerned a common symbolism, has been codified into text books and a standardized notation established. It operates administratively under two legally constituted bodies, the Institute for England and the Faculty for Scotland, with a defined curriculum of study, examination tests, and the power of conferring degrees. The mortality of assured and annuitant lives has been repeatedly subjected to exhaustive investigation and analysis; and has been brought into such shape as to permit of the ready calculation of the monetary value of life contingencies of all kinds. Of all this, and much else, the record stands in the pages of our *Journal*, and the corresponding literature of Scotland. But there were harder problems than these; harder, and of more vital importance to the stability of life assurance business. The rush of activity and expansion which characterized the century brought in its train, as we all know, the perils of inflation, corruption, and reckless ignorance. These left their mark on the life assurance field as elsewhere. Against them, the early actuaries fought the battle for sound valuation bases and adequacy of reserves. Things which now to us seem elementary had to be contended for on first principles by men who are with us no longer, but who worthily discharged this sterling service to their kind. Further, and more practically still, they recognized that the crying abuses which threatened to jeopardize the whole reputation of the business were not to be effectively dealt with otherwise than by law; and they set about the problem of obtaining help by legislation. Of that legislation in its main principles it is no secret that they were, if not "the onlie begetters", at least the inspirers; and now, at the end of nearly half a century, we who succeed them are able to say how its action has purified the air of our business, removed the stigma which had begun to attach to it, and set it firmly upon a road of honourable progress.

In such circumstances, then, and in response to such conditions, the actuary has taken his place among the figures of our national public life. From one great and unavoidable drawback he suffers. The name of his profession is an unfamiliar one. Most of the men whom the Englishman meets in business follow callings the names of which stretch back for centuries, are saturated with history, and call up a simple and unmistakable mental picture. Everybody knows at once what a man means when he declares himself a banker, a doctor, a lawyer, a brewer, or a clergyman. The word actuary, however,

had no such happy power of connotation, and it is to be feared that the atmosphere it has created has in consequence been rather a chilling and discomforting one. On occasions one has heard that there is a charm about the unknown, but this, I think, implies that it be accompanied by a spice of romance; and romance is not exactly what one would think of in connection with our profession. Without this saving grace, the unknown is apt rather to suggest suspicion, if not alarm. I suppose there are few of us here who have not felt a little puzzled, on meeting for the first time someone innocent of the whole subject, and seeking information, with the effort to give concisely a clear idea of what an actuary is. There is, of course, an order of mind to which the problem presents no difficulty. Taking up the other day a book written especially for the "outside men" of life assurance business, by a genial purveyor of such literature, I happened to open on the statement that an actuary is a mathematical acrobat. It was with satisfaction that I found myself connected, however distantly, with a profession the ingenuity of which I had always profoundly admired, but to even the outer courts of which I had never thought to attain.

Seriously, however, it seems as if the actuary has had some ground for disappointment at the way in which his work and functions have often been regarded in quarters which are or should be free from any suspicion of being insufficiently informed. I am thinking of such a phrase as "the actuary must be satisfied," used in a kind of pettish way as if what was proposed was a concession to amiable eccentricity. For reasons which will be mentioned later I think this attitude is much rarer than it was, and is likely to become rarer still. It is mentioned not by way of complaint, but as something which it is well worth our while to try to understand. At its root lies the impression, unavowed but none the less real, that the actuary has taken sweet counsel with formulas, and communed with theorems, till they have distorted the perspective of his outlook upon facts; and that as the Buddhist saint tries to get touch with the infinite by fixing his gaze for a succession of years on one small object in space, so the actuary may let the stream of affairs go by him unheeded, in rapt contemplation of a decimal point.

In face of the record of activity which we have been reviewing, such a notion may well seem unreasonable; but we

shall be very foolish if we pass it by without considering whether we have always and everywhere been guiltless of giving ground for it. When we think this over we see that the temptation indicated is a real one, incidental to our profession on account of the purely abstract type of reasoning on which its principles are based. The remedy is to get, and keep, in touch with fact; and so acquire and attain instinctively that sense which can discriminate between the accidental and the essential when translating theory into practice. The duty of the actuary is not an easy one. His estimates and providings are concerned primarily with the future, and the fairly distant future. Hence it is imperative that they be on the side of safety. We have all recently been taught, in letters of fire, how unsuspected and unforeseen catastrophic change may be. But the results of making abundant provision for the safety of the future may well be disappointing to the expectations of the present: and while in any ordinary business the loss caused by a mistaken financial step will quickly reveal itself, in the affairs with which the actuary is concerned the results may not appear for a generation. Such conditions call for an unflinching integrity, a constant sense of trusteeship for the interests of the future, and a courage which, when once the judgment has arrived at its conclusions, will act upon them consistently and fearlessly.

Such considerations as these can never be too vividly before us, and I think no apology is needed for emphasizing them here. The immediate cause which led to their introduction, however, was a reference to popular misconceptions of the actuary's place and work; and here there has during the last few years been a great and salutary change. It has come about in an interesting fashion, for it has been coincident with the growth in importance of what we may call the public side of actuarial activity. Of that, hitherto, we have said little. The spirit of the epoch in which actuarial science underwent its great development was individualistic. Only by minor indications, slowly but steadily growing, were possibilities of a wider scope indicated—in advocacy, for instance, of more frequent and fuller census returns, and the like. As we all know, there began about ten years ago in the public life of our country a definite attempt at social amelioration by legislative enactment. We are not concerned here, where party politics do not enter, with the controversial side of this movement; which is indeed of the less

importance as differences of opinion chiefly centred round the expediency of the means rather than the desirability of the end. What is certain is that such measures as those for securing Old Age Pensions, National Health and National Unemployment Insurance, filled the atmosphere of public life with new questions and directed public attention in quite a new way towards the actuary.

The fact that schemes of such national importance and financial magnitude were possible, and became possible by the scientific manipulation of masses of statistics, impressed the popular imagination in a way in which nothing connected with the actuary's work had hitherto done. It must often in years past have occurred to many of us that the great life assurance industry, considering the sums of money, national wealth, which it controlled, collected, invested and distributed year by year, had not its relative importance assigned it by the man of average intelligence in his outlook on national finance. The reason is, perhaps, not far to seek. There was no simple and impressive way in which the total magnitude of the figures came automatically into view. From the nature of the case this could not be so. They were discoverable, as totals in a blue-book, by anyone having the courage and patience to extract them. But a blue-book is to the average Englishman a thing of terror, when it is not an object of complete indifference. There is a pathos about these compilations which is, when we reflect on it, irresistible. Containing honest work and solid fact on subjects of all kinds, each of interest easily greater than that of a hundred modern novels, and subject to the gloriously democratic arrangement of being sold by weight, they stand neglected in woe-begone ugliness, untouched except by a small band of specialists. When there is once more time to think about such things, a Government might do worse than consider the possibility of presenting some of them, judiciously edited, in a form and with an arrangement as attractive, say, as that of the "Everyman" series. The experiment might prove worth while. This, however, is a digression. The point is that the policyholder saw, perhaps, the figures of his own company in its annual report, but nothing further. He had no conception of the greatness of the whole. To consider the figures of a national scheme was to realize suddenly how great and important were the interests which actuarial science might control, and how vital to the public were wisdom and competence in those who

should be entrusted with its application to the problems involved.

Upon all that time of social scheming, which looks so far away now, fell without warning the thunderbolt of war. In a moment the entire aspect of things was changed. Sums of money about which, as expenditure on social reform in a year, there had been grave hesitation and debate, were now exceeded in a few weeks by wealth poured forth as the necessary cost of a struggle in which the vital interests of the nation were at stake. That struggle continues. Those conditions surround us still. Our period of retrospect is over. We have now to ask how the fabric reared in the days of peace has met the strain of conflict.

By the courtesy of the representative officers of British institutions transacting life assurance and annuity business in this country, I have been able to obtain some figures which will, I think, be interesting and significant. Since the beginning of the war these institutions have paid, or admitted for payment, war claims amounting to over £7,500,000. They have invested in British Government securities of various kinds upwards of £75,000,000. They have sold or lent to the Treasury, under its schemes A and B, securities of the face value (converting the dollar for this purpose at five to the pound) of over £46,000,000.

In connection with this final item, it must be remembered that before the Treasury schemes had appeared large blocks of American securities had been sold by the life assurance companies in open market, many of which may have passed into the hands of the Treasury, and the sale of which, in any case, effected the same purpose as the Treasury had in view—the maintenance of the rate of exchange. My enquiries as to this have not been such as to enable me to give such exact figures, but I think the total amount may safely be taken (on the same basis of dollar conversion) as not less than £20,000,000.

Looking at these items one by one, let us also bear in mind the conditions of 1815 by way of contrast.

The war claims represent an alleviation of human suffering, and a solace in affliction, of the utmost value. From the nature of the case, they must in the great majority of instances, regarded as individual transactions, represent losses to the companies. That is to say, they must have fallen on comparatively young lives, upon whose policies the premiums received have been small in amount as compared with the sums assured. Further, it may be taken as practically certain that in most cases nothing

beyond the ordinary rate of premium has been paid. For many years before the war, so great was the general confidence in a continuance of peace, or at least in the unlikelihood of our ever having to raise and equip a foreign service army from our civilian population, that according to the general practice of the companies any proposer for life assurance who, being neither in naval or military service, declared that he had no intention of joining either, was granted at the normal premium rate a policy free from all restrictive conditions as to such service; a contract which of course, once issued, bound the company for all time.

The strain of this sudden and heavy mortality is no slight one, but it has been borne without the least disturbance by the companies concerned; and I think general experience will attest that all possible promptitude has been shown in settlements, and everything practicable done to make the receipt of the money by bereaved relatives easy and expeditious.

Now here is a system of mutual help, which has silently and steadily grown during the last century, the result of purely voluntary effort, which has vindicated itself at a time of supreme test, and softened the shock of financial loss by a process of equalization and distribution of burden.

A point of even greater importance is to be made in connection with the second item on our list; the purchases by the assurance companies of Government securities since the outbreak of war. What our National Debt will be when peace returns depends on the duration of the conflict, and as to that he would be a bold man who would venture to prophesy. One thing, however, is sure—that the burden will be a heavy one. Now in facing that burden as a nation, and bearing the taxation it must involve, one of the most important things to secure is a widely diffused holding of the stock. A National Debt held by one class, and paid for by another, is a national danger. Literally, of course, such a state of things could not exist, for taxation must fall in some degree, directly or indirectly, upon the whole community. In practice, however, the exclusion of the poorer classes from the possibility of becoming stockholders would tend in that direction. This was one of the ugliest difficulties of the years that followed 1815. Fortunately there are other and more direct grounds, as we shall see later, of assurance that the position will be very different now. My present purpose, however, is to urge that this holding of stock by the assurance companies is a holding by their assured. It is

a democratic holding. At one remove, the War Loan held by his assurance company is the property and investment of the policyholder. The interest paid on that stock fructifies in his favour. It is an asset to his credit. His efforts to bear his share of the national outlay necessary to pay that interest are directed, not solely to the claims of others, but, *pro tanto*, to his own. Nor is this one of those arguments which, even if true, are too subtle and fanciful for the plain man. When he sees, on the balance sheet of the company which holds his premiums, a large sum of War Stock, he is not likely to forget that anything which would shake the security of that investment would jeopardize his own property. These are conditions which strengthen a people when it is faced with a difficult financial task. They make for national stability and solidarity, things that in such circumstances are of vital importance.

In this connection there is one other fact which I should like to place on record. By the courtesy of Mr. W. A. Platt, late President of the National Conference of Friendly Societies, I learn that the Friendly Societies of the country have invested in War Loan Securities about £1,700,000. There are difficulties as to getting exact figures, and this amount must be taken as approximate only, but it seems to me to be deeply significant in the light of what has been said as to the value of a partnership of all classes of the community in our national obligations.

The negotiations between the assurance companies and the Treasury present another aspect of the possibilities for national help in time of crisis which these accumulated popular savings (for such they are) afford. They can be of service internationally. The companies, primarily interested in investing the money of their policyholders on the most favourable terms, have long since found that this could only be done by judiciously widening the investment field. As a consequence they became large holders of high-class American securities. When the outbreak of hostilities simultaneously reduced our capacity for export and made us importers on an unprecedented scale of war material from America, the result on the rate of exchange was of course heavily felt. By various expedients the difficulty had to be met, and among these one of the most effective was to make use of the American securities held by the assurance companies, which were ready to hand. The process, as the figures above quoted show, was already beginning to work automatically—

in the attraction offered by the influence of the rate of exchange on prices; but co-operation between the companies and the Treasury made it possible to develop an ordered scheme as part of a general plan of action.

It has also to be borne in mind that the companies have subscribed substantially to loans issued by the country's Allies and by our Colonies during the war. I have not collected statistics on this head, because the main object in view was to show how the subscriptions to our own debt diffused its holding throughout the community and broadened its base. The support of such loans, however, is another genuine and important factor in the strength for national purposes revealed in the life assurance business of the country by the exigencies of war.

Summing up then this recital of services rendered; the war claims paid, the war stock purchased, the help given through the Treasury for international finance, and the support of the loans of our Allies and Colonies; it may surely be said that the life assurance business of our nation, which we have claimed as the outstanding achievement of actuarial science during the century, has triumphantly justified itself under the test and strain of the country's need.

When referring a few minutes ago to the importance of a widely representative holding of the National Debt, it was remarked that for reasons other than those immediately under discussion, there was little cause to fear failure in that respect. The reference was to the issue of the now famous War Savings Certificates. It is the more appropriate that special notice should be taken of these, as it is no secret that actuarial advice has been active both in the inception and the working of the scheme. It is a direct attempt to enlist the sympathies and support of the industrial section of the community in raising the necessary funds for the prosecution of the war. We have recently been told in the House of Commons that up to the end of October 1916, approximately 44,500,000 £1 Certificates had been issued—the scheme having only been inaugurated on 22 February of the same year. This, therefore, represents the results of just over eight months' working. For the first three and a half months investment in these Certificates was forbidden to persons with annual incomes exceeding £300, and during that period the issue was approximately 4,275,000 £1 Certificates. Then the income limit was removed, the only restriction remaining being the maximum holding per

individual of 500 £1 Certificates. This widening of the net had a great effect on the total. Since 10 June 1916, there have been about fifteen and a quarter millions of £1 Certificates issued in the maximum batches of £500—a clear proof that classes other than that of the industrial worker have availed themselves of the scheme. Notwithstanding this, however, it is evident that the small investor is still steadily attracted, for in the same period just about an equal number of the certificates were issued in sets of from 1 to 25. As a matter of fact I think those best qualified to know would tell us that the scheme is making well sustained headway where its extension is most to be desired; the untiring and well considered efforts of the National War Savings Committee are daily bringing the matter home to our democracy, with the result that each week shows a substantial increase in the number of certificates sold. This is the more gratifying as the Savings Bank deposits for the year, so far from showing a diminution as the result of this new channel of investment, have increased by four and a half millions sterling. We know that just now our people are prosperous. We know too that much of that prosperity is factitious, and must have an end. We fear that they may not realize this, and may make imprudent use of the days of plenty; and that fear is often very loudly expressed. Let us at all events welcome any evidence which goes to show that there is some instinct of prudence, and give all the possible help we can to the direction of that instinct into this best of all paths—the taking up of part of the national security. "Certificates of loyalty" these £1 holdings have been well called by one of our own number. Loyalty can have no better employment than in advocating and helping their success. Every holder of a War Savings Certificate is a partner in his country's effort, and a contributor towards its victory.

Among the practical questions of present interest in which our profession is specially interested one of the most important is that of life office valuations. Here as much as anywhere in the national finance the effect of the war has been felt. It caused an immediate and very formidable increase in a difficulty which had been existent for many years—the persistent depreciation in security values. So prolonged and unvaried had that downward movement been that, as we all remember, the air was full of suggestions of various kinds as to how the situation might be met by some special measures; such, for instance, as some modifications of the requirements for valuation

of investments in the balance sheet, or a change in the rate of interest employed in the valuation of the liabilities. In the latter connection it was pointed out with considerable force that the real rate of interest realized, if securities were written down to market values, was substantially higher than that which had been procurable when the margin between it and the valuation rate had been held adequate for safety. If a company when it realized all round, after deduction of income tax, four per-cent on its funds, was justified in valuing at three per-cent, when the realized rate became four and a half per-cent might not the valuation rate rise by a half per-cent also? There were grave objections to any practicable suggestion, and the whole subject was recognized as one of extreme perplexity, which yet might some day demand a decision. Striking in on all this, the war transformed the whole scene. Security values fell to unknown depths. The rate of interest rose rapidly, but alongside of it in grim proportions the income tax rose too. The problem of valuation became more difficult than ever, but its difficulties were of a more serious kind. Conditions undreamed of and unprecedented had arisen. In these circumstances it must stand to the undying credit of the companies that they had the courage to adopt the motto, "Stability first; profit second." As would of course happen, the conditions have affected in varying degrees, according to circumstances, the offices whose valuations have fallen due since the crisis began; but the motto just quoted has determined, as we may be sure it will continue to determine, the line of action. No company which has felt it prudent in the circumstances to reduce, postpone or pass its bonus, will suffer for it in the judgment of any wise or thoughtful man. To take in sail in a storm, to consolidate resources in a time of uncertainty and danger, are steps the vindication of which may safely be left to the sure and irreversible verdict of time.

Mention was made just now of the income tax; and this introduces the second present-day question which calls for notice. I propose to say a few words only with reference to this question. We know that the subject has been one of perpetual debate between the companies and the collectors of the Inland Revenue. As we are promised, however, a commission of enquiry into the whole subject, and as it is admitted to present, in its existing condition, many anomalies, we may take the opportunity to make a brief reference to the principles involved. The present

circumstances are of course those of emergency. The action of economic law here is rather curious. Money is in demand. It is imperative that the Government should have it to meet war outlay. In consequence its value is high. On the other hand, the Government taxes interest, as such. This tax, as a source of revenue, is heavily increased. So far as the recipient of interest is concerned, these two tendencies cancel one another. He gets a high rate, but subject to a high tax, and is very much where he was before. Now it is easy to be unreasonable, even unpatriotic, in this matter. Sir Charles Lucas, in a memorial notice of that great Englishman, the last survivor of a brilliant band, John Llewellyn Davies, quotes the testimony of a friend that he was the only man he ever met who felt "a positive pleasure" in paying rates and taxes. We smile at this, but may learn something from it. Many men talk nonsense about taxation being robbery until they come to believe it. Taxation is a touchstone of citizenship. That we fully recognize; but the point as regards the life assurance companies lies in the unique part which interest plays in their business. It is necessary, according to the assumptions on which that business is built, to enable them to fulfil their contracts. The Government has a case in point,—the War Savings Certificate which we were recently discussing. That is really a capital redemption policy for a five years' term at a single premium on a five per-cent basis. What has the Government done there? It has to fulfil its contract. It makes the security free of income tax. We do not ask for this. Nor should we ask, if I may venture an opinion, to be taxed on profits. One reason I would give for this is that "profits" as we understand them hardly correspond to the strict meaning of that term. A considerable portion of those "profits" have been paid for, by premium loadings. We should also have possible complications and difficulties arising out of the varied systems of valuation and distribution adopted by different companies. We have, I think, a better and fairer basis in adhering to the principle of payment on interest income, but with some discrimination in the proportion of that income made subject to tax. Logically, it might be the surplus by which the interest realized exceeds an assumed average valuation rate. I do not, however, make any definite suggestion, for the subject abounds with difficulty. All that I have in view is to try to indicate the *rationale* of our claim for reconsideration.

An outstanding event of actuarial interest during the past year has been the work of the Departmental Committee on Approved Society Finance and Administration which has been appointed to consider questions arising out of the working of the Health section of the National Insurance Act. It is a carefully selected and thoroughly representative body which under the able chairmanship of Sir Gerald Ryan has worked with an energy, a devotion, and a thoroughness beyond all praise. Setting apart regularly for its labours not less than two days in each week it has received and considered a very large body of evidence, and has published an interim Report in May last. That is so ably summarized in our *Journal* that there is no need here to attempt to recapitulate its contents, apart from the fact that within the limits of time at our disposal no adequate account of them could be given. It is, however, a document of great interest and value, which will well repay close study. From the first it has been recognized that the National Health Insurance Scheme must from the nature of the case be looked upon as to a substantial extent experimental. Experience alone could teach some of the lessons which must be learned before the instrument could be perfected. Experience has revealed two outstanding difficulties; the excess of actual over assumed sickness rates among female lives, and the disadvantageous position occupied by approved societies which derive their membership wholly or mainly from those engaged in exceptionally unhealthy trades. The provisions of the Act are also considered insufficiently elastic to cover temporary periods of excessive sickness or other adverse contingencies. The Report recommends meeting these difficulties by a reduction of the Sinking Fund, and by setting up, with the income thus freed, special Funds for a period of ten years,—experimentally. This will have the effect of prolonging for six years the operation of the Sinking Fund. This is the general result, but the real force of the Report is only to be appreciated by careful perusal. The minute and impartial attention bestowed on every detail; the strict adherence to equity among the several classes of beneficiaries; the resource and ingenuity shown in handling the position; give confidence that the work is in capable hands and that the history of this great national enterprise will be one of success.

Within the last few days, and since the foregoing words were written, the full Report of the Committee has appeared. Its length, and the short time since its publication, make any worthy

or adequate notice of it here impossible. All that can be done is to record its issue, and to bespeak for it the careful study it demands. It makes further important recommendations, some of which have been summarized in the daily press. It bears throughout evidence of the qualities which have been mentioned as characterizing the interim Report; following the same line of making past experience a guide to the operations of the future.

That mention of the future brings us to the last, and by no means easiest, part of our survey to-night. When it will happen is given to none to know, but the time is certainly approaching nearer every day when the war will cease, and we shall be confronted with the problem of rebuilding the social fabric and repairing the ravages of destruction. How shall we set about it? As those who have learned nothing and forgotten nothing, or as those whom this unparalleled experience has made wiser than they were?

The financial position up to the end of March next, assuming the war to continue so long, has been forecast for us by various authorities. The Chancellor of the Exchequer in his speech of 10 August last, estimated that under these conditions the National Debt would by 31 March amount to £3,440,000,000, and the advances to our Allies and Dominions to £800,000,000, leaving us with a net indebtedness of £2,640,000,000. Against this, he said, we should be collecting a revenue in one year equal to one-fifth of the net debt—say roughly £500,000,000. Besides this he set estimates of the total annual income and capital wealth of the nation amounting respectively to 2,600 millions and 15,000 millions sterling. Sir George Paish, addressing the Statistical Society in March last, and approaching the subject in rather a different way, arrives at an estimated net war indebtedness, in March 1917, of 2,247 millions. As regards the national income, he places it, as inflated by the war, at 3,000 millions, but regards its normal amount as 2,400 millions; and estimates the national capital at 17,000 millions sterling. No one who has looked into this subject of estimating the national income and capital can fail to have been impressed by the difficulties such a task presents. Dr. J. C. Stamp in his great book "British Incomes and Property" gives a valuable table showing the various estimates which have been made in past years. He supports on the whole Sir L. Chiozza Money's estimates for 1914 of 2,100 millions income and 16,000 millions capital, and quotes for 1812 (the nearest we can get to 1815) the

estimates of Colquhoun; 431 millions income and 2,700 millions capital. It seems probable that the 1914 estimates, especially that of income, are unduly low for 1917, even deducting the war inflation; but if we accept them, and assume Colquhoun's 1812 figures as good for 1815, we can get a rough view of the comparative position. The National Debt at the end of the Napoleonic wars was 900 millions. If we take the Chancellor's estimate of a net debt of 2,640 millions in 1917, it is clear that we should stand, as compared with 1815, in a position twice as favourable in relation to capital, and one and four-fifths times as favourable in relation to income.

Interesting as these estimates and forecasts are, there is so much about them all which must be speculative that they cannot be made the ground for positive inferences. The great determining factor, the duration of the war, remains an unknown quantity, and that fact makes every calculation a guess. What we may take as certain is that when the end comes the nation will find itself faced with financial responsibilities dwarfing into insignificance those of any past period of its history. It will face them with a strength of resource greater, both relatively and absolutely, than it has ever known before. The problem can be triumphantly solved without friction or discord, if (and this condition is essential) it be met along the line of facing things as they are, remembering that the war has changed everything, that there is no virtue in any former tradition unless in the changed conditions it can hold its own by virtue of its inherent reasonableness, and that we have to build up a new social order which in many ways must differ from that of the past.

If we are to come through our difficulties, the productivity of the nation must be increased. In saying this one is of course thinking of the pre-war standard. Since the conflict began, that productivity has risen to a level which no one would have deemed possible. This has happened because in presence of a common peril we have thrown away old restrictions and obstacles and worked as those who have one end in view. If that attitude can be continued, we shall have nothing to fear. The chief prospective difficulty in the way lies in the relations between labour and capital.

The very mention of these words suggests sharp bygone controversies. Let us put that spirit aside, and look at the question from the standpoint of that great solver of strife,

history. What was the course of reconstruction after 1815? It began badly. For many years previously the policy of enclosure of common land, steadily pursued, had, while it certainly promoted agriculture, reduced one class of the community, the cottagers or small holders, to abject poverty. When the factitious prosperity caused by war prices ceased, therefore, and the pinch came, there was already one section of the populace in the extremity of distress. Then came, breaking up and transforming social conditions on all sides, the enormous advance in the industrial revolution, and the factory system. We were not prepared for the suddenness or the scale of these changes. It is difficult not to feel that if the rate of material progress had been half as fast, the sum of human happiness would have been greater, and that of human misery less. But there was no checking the natural expansion. We became a manufacturing people. Men and women flocked into the towns. Population trod close on the limits of subsistence. Politically, the spirit of the time was individualist. The state had to maintain order: to keep the ring for the survival of the fittest. So came about the state of things which Carlyle has described as "anarchy plus a policeman." The workers became numerous and strong enough to make their united power formidable. The principle of "collective bargaining" was established: with the result that we have now some four millions of operatives (and the bulk of these, remember, among the most intelligent, competent, and self-respecting of their kind), members of Trade Unions.

So the development went on, in more or less haphazard fashion, working itself out with what I think will seem to us all some day a considerable waste of force and energy, and a good deal of preventible suffering to all classes. Meanwhile, however, a sense of deep dissatisfaction, not always articulate, had been growing. The rigid individualism of early Victorian days was dead. If set to find a phrase in which briefly to express the slowly developing ideal which began to make much party strife seem trivial, one might choose the words "The nation a family." They have the defects and limitations which any such condensed description of a complex thing must have, but the root idea is of a community whose interests shall be treated, not in rhetoric but in fact, as co-operant, not antagonistic. Then came, as a bolt from the blue, the war; which, whatever its horror and tragedy, will be ever memorable for some of us as having at a step brought us nearer this ideal than we might otherwise have come

for many years. Shall it hereafter be said that we have been thus fused into one only in presence of a common peril, and that when it is removed our old divisions will reappear? Is there then no other peril, at least as deadly if not so obvious, to which in our days of peace we had drifted far too near, and which waits upon a disintegrated people? "A house divided against itself cannot stand."

Fortunately, we have cause to hope for better things. In the very valuable little book published under the auspices of the British Association, entitled "Labour, Finance and the War", the Committee of that Association which examined the question of industrial unrest, making a recommendation that there be for the future regular opportunities of conference between employers and employed, not as emergency measures when trouble has occurred, but as part of a machinery preventing its occurrence, describes the proposal as having had the approval of well-known representatives both of capital and labour. Professor A. W. Kirkaldy, as Chairman of the Committee and Editor of the volume, emphasizes this in words which I ask permission to quote. "The attitude of all, supplemented by that of other "representative men interviewed in the course of the inquiry, "strengthens the hope that we may be on the threshold of a new "era. If only the attitude taken up under the stress of war be "maintained for a few years a peaceful revolution will result, and "we shall wonder why we did not take the obvious steps sooner."

Is all this out of place in a discussion of topics of special interest to us as actuaries? I venture to think not. When we consider it closely, we find that the principle underlying our science, the assessment of hazards so that they may be met by an average continuous effort and thus deprived of their power to produce individual catastrophe, has very much in common with any effort to bind a community together in one. A friend of mine used to call insurance "the socialism of the practical man"; and those of us who had the pleasure of hearing or reading Mr. Armstrong's recent Presidential Address to the Insurance Institute, will remember how eloquently he emphasized the idea underlying such a phrase. Fortunately, we have a striking instance ready to our hand. Discussion and negotiation respecting the Health Section of the National Insurance Act have naturally rather overshadowed the history of its second section, which dealt with Unemployment Insurance. All who read the first Report under that Section, published in 1913, must

have been impressed by the amount achieved in the way of organization and the smoothness of working secured, as well as by the assistance given through the trade unions. over half-a-million employees arranging their operations under the Act through such societies. Through the good offices of my friend Sir Alfred Watson and by the courtesy of the Board of Trade, I am able to say that the total accumulated amount of the Fund at the present time is £7.475.000; and that the total amount paid to workpeople in unemployment benefit from the beginning is nearly £1.230.000. The number of insured workpeople (other than the munition workers whom recent legislation has included, and figures concerning whom are not yet available) is estimated at rather over 2,100,000. Here then, in view of the depression and unemployment we may have to face in some of the years succeeding the war, is a store laid by which will be of the utmost value. So far as the trades covered by this legislation are concerned, there is provision towards the day of trouble gained by a steady process of contribution which has inflicted no hardship.

As we look into the future, with the hope that the great ideal of a nation bound together by ties of sympathy and mutual understanding may more and more prevail. possibilities of the extension of this principle of common protection seem to increase and multiply. There is probably no ordinary hazard of life which cannot be thus dealt with. subject to one great preliminary condition. We want the facts. Given these, so that the true bearings of the problem may be set out. our science is daily becoming more competent to attempt its solution. The scientific method of dealing with statistics is a subject to which some of the most brilliant minds in our profession are increasingly devoted. So I would venture again to urge. as one of the *desiderata* of reconstruction. that Government Statistical Department for which Professor Bowley some years since put in a plea. The under-estimate of female sickness rates in the National Health Insurance scheme. to which reference has already been made, was entirely due to the absence of statistical data. We ask for full and abundant facts, elicited by those who have statutory authority to support them in their requisition.

We have said that the great achievement of actuarial science during the last century has been the construction of the fabric of life assurance. There the main principles are laid down, the practice is established. To its conduct the skilled labour of the actuary is and will remain essential, but it is not likely

to assume any radically new form. May it be that in the epoch near whose threshold we now stand, one of reshaping and remaking the national life, the correspondingly characteristic part that actuarial science is to play will lie in its work along the multiplying paths of public service which will open before it?

Upon such a note of hope I would address myself especially in conclusion to the younger members of our profession. It is difficult on the present occasion to do so without feeling a certain presumption. Upon the young has fallen the tragic burden of the nation's ordeal, and in comparison with what they are doing and have done, all else for the time seems overshadowed. Many who would in ordinary circumstances have been with us to-night are risking their lives for their country on the field of battle. Some will be with us no more. But a time is approaching when this hard path will have been traversed to the end; and to those who then stand at the outset of an actuarial career the opportunity of that coming day holds out a hand of invitation. "Peace hath her victories, no less renown'd than war." The building up of a social order cemented by mutual confidence and organized for mutual help is a task in which it will be no mean honour to have borne a part. To the accomplishment of such ends, the need of scientific guidance in such questions as you study is more and more widely acknowledged. Of this I may just give one significant piece of evidence. Our larger Friendly Societies are appreciating more and more the necessity of seeking the advice of actuaries in the conduct of their affairs, and the great majority of the current valuations of Friendly Societies have been placed in the hands of members of the Institute. In that direction it is instinctively felt that safety lies. This is chiefly interesting as being typical of the general attitude of the public mind, in its recognition of the need for the help of actuarial science in the working out of the social problems of the future. And to what worthier vocation could men be summoned? To direct, with scientific foresight, enterprise which has for its end the increase of well-being and the diminution of poverty in a nation; to assist in building up protection against disaster, and the security which gives peace of mind; to lay down the lines along which common effort may redound to the good of all, and the prosperity of a people be unshaken in its fields and in its homes;

"These are imperial works, and worthy kings."

Frequency Curves and Correlation. Addendum (with Diagram) and Errata. By W. Palin Elderton, F.I.A.

Since 1906, when "Frequency Curves and Correlation" was published, further work has been done on the subject, and the following pages are put forward with the idea of bringing the book more up to date.

There is a warning in connection with probable errors dealt with in Chapter VIII which should be borne in mind in all statistical work. It is that we are not justified in assuming that the normal curve of error describes the distribution of errors in all cases. When the number of cases is large it is generally sufficiently accurate, but in many cases it is unsuitable and in some cases may lead to wrong conclusions. This may be inferred from the many peculiar shapes of frequency distributions found in general curve fitting and from the fact that actual samples of correlated material give all these extreme forms and show that the criticism is a practical one which must be borne in mind in practical work.

The study of deviations in these cases is still being developed; it is of some importance in actuarial work where the deviations with which we are concerned arise from a distribution which is by no means normal, *e.g.* $(p+q)^n$ where q the probability of dying in a year is a very small quantity.

A list of mistakes in "Frequency Curves and Correlation" is also given. I apologize for these mistakes and thank the many friends who have helped me by calling my attention to them. I am afraid a few mistakes may still lurk undetected in the book; if so, I can only hope that they will cause little inconvenience to readers.

I.—Uncommon Frequency Types.

In the course of statistical work a distribution is sometimes found which appears different in its algebraic form from the usual types, but can nevertheless be described accurately by those types. An example which will give an indication of the kind of case we have in mind is a distribution arising from recording the number of sequences in coin-tossing or dice-throwing experiments: the distribution is a geometric progression and this, although a well-known result in probability, is not at first sight a necessary form of Pearson's series of frequency curves. The expression, however, for his Type III is $e^{-\gamma x}\left(1+\frac{x}{a}\right)^{p}$ and

if p be put equal to zero, we obtain the exponential $e^{-\gamma c}$ which gives the series we want. This possible limit was casually pointed out by me in "Frequency Curves and Correlation," but I did not investigate generally the circumstances in which this happened, nor did I appreciate that other interesting special cases might arise with other types. A recent paper by Prof. Pearson deals with various special cases which are so different from the curves to which we are accustomed that he has dealt with them as new types. These types arise in certain limiting cases of Types I, II and VI, and give straight lines, curves starting with an infinite and ending with a finite ordinate, two separated blocks of frequency, and curves starting at a finite ordinate and ending at zero either at a finite point or at infinity: among these last is, of course, the exponential to which we have already referred.

Before turning to the expressions for these new types it may be useful to give a table of various peculiar distributions that have been obtained from insurance and other material.

Examples of uncommon Frequency Types.

469	45	119	4,165	33	68
186	38	100	2,028	53	24
166	46	86	982	65	17
134	53	75	480	81	14
122	43	61	266	101	12
112	38	50	132	131	11
...	49	39	71	186	10
...	41	27	36	350	10
...	44	22	17	...	10
...	52	12	9	...	11
...	...	3	2	...	12
...	...	...	1	...	20
...	...	...	1	...	...
...	...	...	1	...	...
...	...	...	1	...	...
1,189	449	594	8,192	1,000	219

This table shows distributions which at first sight have little likeness to the frequency curves with which we are familiar; they can, however, be fitted with reasonable accuracy by Pearson's generalized curve. Some of them are rather like the J-shaped curve, of which examples are given on pp. 66, 104 of "Frequency Curves and Correlation", but others have little resemblance even to this extreme type. The table includes (col. 6) areas of a U-shaped curve which is rare; in fact, I have

not succeeded in finding a suitable distribution of this shape among actuarial statistics, but such a distribution might occur among terminations (including withdrawals) in term policies of ten years, say, or similar endowment assurances.

It will now be convenient to take each curve in order and give the formulæ to enable it to be fitted to the statistics, the circumstances in which it arises (*i.e.*, the criterion), and numerical examples. It may be remarked that the types can be easily picked out from a diagram given in Prof. Pearson's paper; this diagram sets out for all possible values of β_1 and β_2 the type that should be used.

Type VIII.

$$y = y_0\left(1 + \frac{x}{a}\right)^{-m}$$

Range from an infinite ordinate at $-a$ to a finite ordinate, y_0, at 0.

m is found from the solution of

$$m^3(4-\beta_1) + m^2(9\beta_1 - 12) - 24\beta_1 m + 16\beta_1 = 0$$

and must be not <0 or >1.

$$a = \pm\sigma(2-m)\sqrt{\left(\frac{3-m}{1-m}\right)}$$

$$y_0 = N(1-m) \div a$$

The distance of the mean from $x = -a$ is $a(1-m)\div(2-m)$. When μ_3 is positive a is negative.

The curve is a special case of Type I when m_2 is zero, that is, when

$$r - 2 = r(r+2)\sqrt{\{\beta_1 \div [\beta_1(r+2)^2 + 16(r+1)]\}}$$

where $$r = 6(\beta_2 - \beta_1 - 1) \div (6 + 3\beta_1 - 2\beta_2)$$

that is when

$$\frac{(4\beta_2 - 3\beta_1)(10\beta_2 - 12\beta_1 - 18)^2 - \beta_1(\beta_2+3)^2(8\beta_2 - 9\beta_1 - 12)}{(3\beta_1 - 2\beta_2 + 6)\{\beta_1(\beta_2+3)^2 + 4\beta_1(4\beta_2 - 3\beta_1)(3\beta_1 - 2\beta_2 + 6)\}} \text{ or } \lambda \text{ say}$$

is zero.

The criteria for Type VIII can be reduced to (1) special case of Type I, (2) $\lambda = 0$, (3) $5\beta_2 - 6\beta_1 - 9$ is negative. It may be added that $24\beta_2 - 27\beta_1 - 38$ is small; theoretically positive.

If $\beta_1 = 0$ an interesting special case arises, in which $m = 0$,

and the curve becomes a horizontal line, which is also the limit of Types IX and XII.

The solution of the cubic for m gives trouble. m can also be found from $m=-2(5\beta_2-6\beta_1-9)\div(3\beta_1-2\beta_2+6)$, and though this involves β_2 it should theoretically give the same value of m as the cubic. As the criterion is not exactly reached in practice the two results differ, and it seems preferable to find m from the cubic by using

$$m=\frac{24\beta_1-\sqrt{\{24^2\beta_1^2-64\beta_1(m'(4-\beta_1)+9\beta_1-12)\}}}{2(m'(4-\beta_1)+9\beta_1-12)}$$

where m' is found from the expression in β_1 and β_2 given above or by some other trial method.

An alternative is to find from the criteria or from the diagram at the end of Pearson's paper the value of β_2 which is the consequence of the particular value of β_1 when a Type VIII curve occurs, and use this theoretical value in finding m instead of the β_2 given by the actual statistics.

*Example.**

Frequency	Graduation (1)	Graduation (2)
469	437	436
186	222	209
166	165	161
134	136	141
122	120	127
112	109	115
1,189	1,189	1,189

The mean is ·6518 of an interval after the centre of the 186 group. The constants were

μ_2	2·986
μ_3	3·295
μ_4	18·252
β_1	·408
β_2	2·047
λ	$-$·05
$5\beta_2-6\beta_1-9$	negative. Hence Type VIII can be used.
m	·500
a	$-$5·797
y_0	102·6

* In the numerical work, with a few exceptions, I used more figures than are shown.

The curve runs from ·277 before the middle of the first to ·02 after the end of the last group. The graduation is shown (No. 1) above.

The areas can be calculated by the expression

$$y_{-r} \times (a-r) \div (1-m)$$

which gives the area of the remainder from $-r$ to $-a$. In the particular case the range could be fixed as 6 as the data related to six months' experience of maturities among endowment assurances, and remembering that the mean is $a(1-m) \div (2-m)$ we found

$$m \qquad ·439$$
$$y_0 \qquad 111·1$$

The areas resulting are given in graduation (2). The following table gives the calculation of the areas in this case. The equation to the curve is

$$y = 111·1\left(1-\frac{x}{6}\right)^{-·439}$$

with range from 0 to 6, a being negative because μ_3 is positive.

x	$1-x/6$	Colog (2)	(3) × m	(4) + log 111·1 *i.e.*, y_x	Log $\frac{y_x}{1-m}$	Antilog (6)	Remainder of range	(7) × (8)	Area required
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	...	...	...	...	...	...	6	1,189	115
1	·8333	·0792	·0348	·0804	2·3318	214·7	5	1,074	127
2	·6667	·1761	·0774	·1230	2·3744	236·8	4	947	141
3	·5000	·3010	·1323	·1779	2·4293	268·7	3	806	161
4	·3333	·4815	·2118	·2574	2·5088	322·7	2	645	209
5	·1667	·7781	·3419	·3875	2·6389	435·5	1	436	436

Some of the columns can be dispensed with: they are shown in detail to make the method clear.

Both graduations are reasonably close to the facts.

An example of the limiting case will be found in the following statistics:

No.	Frequency	Graduation	Theoretical
1	45	42	45
2	38	45	45
3	46	45	45
4	53	45	45
5	43	45	45
6	38	45	45
7	49	45	45
8	41	45	45
9	44	45	45
10	52	48	45
...	449	450	450

The mean is ·57 after the middle of the 5 group; the moments are

$$\begin{array}{ll} \mu_2 & 8{\cdot}374 \\ \mu_3 & {\cdot}026 \\ \mu_4 & 124{\cdot}46 \\ \beta_1 & {\cdot}011 \\ \beta_2 & 1{\cdot}78 \end{array}$$

Hence
$$y = 449 \div 2\sqrt{3\mu_2}$$
$$= 44{\cdot}8$$

The range is from ·57 to 10·43.

The series was found by summing in tens the last figure of Carlisle $3\frac{1}{2}$ per-cent Table of A_x and the mean should be at 5·5 theoretically instead of 5·57 and y should be 45. The range should be ·5 to 10·5. The example is interesting as showing how the Pearson-curves graduate in an extreme case. The "graduation" and theoretical results are shown. In the "graduation" decimals have been neglected.

Type IX.

$$y = y_0\left(1 + \frac{x}{a}\right)^m$$

Range from $x = -a$ where $y = 0$ to $x = 0$ where $y = y_0$

$$a = \pm\sigma(m+2)\sqrt{\left(\frac{m+3}{m+1}\right)}$$

m is found by solving

$$m^3(\beta_1-4)+m^2(9\beta_1-12)+24m\beta_1+16\beta_1=0$$

$$y=\mathrm{N}(m+1)\div a$$

The distance of the mean from $x=-a$ is $a(m+1)\div(m+2)$.

As in Type VIII the value of m can be found by simplifying the cubic into a quadratic, or by the other method indicated.

The criteria are reached through the same equation as those for Type VIII, and can be reduced to (1) special case of Type I, (2) $\lambda=0$, (3) $5\beta_2-6\beta_1-9$ is positive, (4) $2\beta_2-3\beta_1-6$ is negative.

If $\beta_2=2{\cdot}4$ and $\beta_1={\cdot}32$ the curve becomes a sloping line

$$y=\frac{\sqrt{2}\mathrm{N}}{3\sigma}\left(1+\frac{x}{3\sqrt{2}\sigma}\right)$$

If $\beta_1=0$ we reach a horizontal line as the limit, while if $\beta_2=9$ and $\beta_1=4$, we have the other limit of Type IX, and find the exponential series (Type X).

Example.

Duration	Exposed to Risk in Annuity Experience	Type IX	Frequency Line
0	119	118	108
1	100	98	97
2	86	85	86
3	75	74	76
4	61	63	65
5	50	52	54
6	39	41	43
7	27	30	32
8	22	20	22
9	12	11	11
10	3	2	0
...	594	594	594

The mean is at 2·909 assuming the exposed to risk to be an ordinate at the duration or an area from $n-\frac{1}{2}$ to $n+\frac{1}{2}$.

μ_2	6·27
μ_3	10·99
μ_4	102·50
β_1	·490
β_2	2·606
$5\beta_2-6\beta_1-9$	positive
$2\beta_2-3\beta_1-6$	negative

The curve is not far from Type IX, and if β_1 had been ·32 and β_2 2·4, we should have reached a straight line $y=111{\cdot}8\left(1-\frac{x}{10{\cdot}63}\right)$ with range from $-{\cdot}6$ to 10·0 and obtained the graduation shown. The whole area $-{\cdot}6$ to $+{\cdot}5$ is taken as the frequency for duration 0. Using Type IX, the following constants are reached:

m	1·123
a	$-10{\cdot}913$
y_0	115·54

The curve runs from $-{\cdot}586$ to 10·3275. The 118 in the first group has been taken as the area from $-{\cdot}586$ to $+{\cdot}5$. Theoretically there cannot be an exposed before duration $-{\cdot}5$, but as we are merely giving an example of fitting a curve to a series of numbers this need not concern us. The difficulty could be met by fitting a system of ordinates or by assuming a starting point for the curve.

If m happens to be less than unity the shape of the curve is somewhat different, *e.g.*, if $y=100\left(1+\frac{x}{10}\right)^{{\cdot}25}$ we have the following ordinates:

100, 98, 95, 91, 88, 84, 79, 74, 67, 56, 0.

The actual deaths in a select mortality experience may take this form, but the shape of the curve will be less flat at the start, *e.g.*, in the recent American Medico-Actuarial experience age group 30–34.

Type X.

$$y=\frac{N}{\sigma}e^{-x/\sigma}$$

Range from 0 to ∞.

Distance of origin from the mean is σ.

The curve is a special case of Type III when $\gamma a=0$, that is when $\beta_1=4$.

The condition for Type III is better stated by $2\beta_2 = 6 + 3\beta_1$ than by $\kappa = \infty$ as mentioned in Frequency Curves, p. 50, &c. Hence the exponential form is given by $\beta_1 = 4$ and $\beta_2 = 9$. The curve is also the limit of Types IX and XI.

Example.

	Frequency	Graduation	Theoretical
1	4,165	4,132	4,096
2	2,028	2,016	2,048
3	982	1,015	1,024
4	480	511	512
5	266	257	256
6	132	130	128
7	71	65	64
8	36	33	32
9	17	17	16
10	9	8	8
11	2	4	4
12	1	2	2
13	1	1	1
14	1	1	1
15	1	...	...
...	8,192	8,192	8,192

The unadjusted mean is 2·0087.

μ_2	2·045
μ_3	6·290
μ_4	39·720
β_1	4·629
β_2	9·502
σ	1·43

When the curve is an exponential the moments and mean require adjustment, but the Sheppard high contact adjustments are, of course, unsuitable. If the curve starts at the beginning of the first group, I think that the mean is overstated when μ_3 is positive by $\frac{1}{12\sigma}$ approximately, and the second moment about the true mean is understated by $\frac{1}{12}$ approximately.*

* The adjustments for the moments for J and U-shaped curves call for consideration. I think something might be done by assuming an exponential form. But I have not had an opportunity of going sufficiently into the point. It is clear that unadjusted moments will not always give entirely satisfactory results. The student should also remember that when the β's are large their probable errors are large too.

Making use of the adjustments the mean is now 1·934, μ_2 is 2·123, and σ is 1·457.

The statistics relate to sequences in coin-tossing and the theoretical figures are added. In the statistics as published the sequence of 11, 12, &c., were 2, 1, 0, 1, 0, 2. In calculating the graduated areas of the curve it is useful to remember that the area from a to b is $(y_a - y_b)\sigma$.

It is interesting to notice how the "graduation" keeps closer to the frequency than the theoretical result.

I give as a second example the following series based on cricket scores known to start at the beginning of the first group:

Score ...	0–19	20–	40–	60–	80–	100–	120–	140–	160–
Series ...	64	34	18	9	6	3	3	0	0
Graduation	64	34	18	10	5	3	1	1	1

The ratio of each term to the preceding is ·54, and the graduation is almost exact. Owing, however, to the 3 at the group 120, the moments give a criterion considerably removed from the theoretical $\beta_1 = 4$, $\beta_1 = 9$.

If we had assumed the start of the curve in the previous example, we should have reproduced the theoretical result almost exactly.

Type XI.

$$y = y_0 x^{-m}$$

Range from $x = b$ where $y = y_0 b^{-m}$ to $x = \infty$ where $y = 0$.

m is found from

$$m^3(4-\beta_1) + m^2(9\beta_1 - 12) - 24\beta_1 m + 16\beta_1 = 0$$

$$b = \pm\sigma(m-2)\sqrt{\left\{\frac{m-3}{m-1}\right\}}$$

$$y_0 = \mathrm{N}b^{m-1}(m-1)$$

The distance of mean from origin is $b(m-1)\div(m-2)$.

As in Type VIII m can be found by simplifying the cubic into a quadratic or by the other method indicated.

m may have any value from 5 to ∞, but in practice its value is not less than 9.

The curve is a special case of Type VI when $q_2=0$.

The criteria can be expressed as (1) special case of Type VI, (2) $\lambda=0$, (3) $2\beta_2-3\beta_1-6$ is positive.

Example.

Duration	Withdrawals	Graduation by XI
0	165	183
1	65	53·9
2	23	32·6
3	32	20·0
4	13	12·4
5	8	7·6
6	1	4·9
7	6	3·1
8	3	1·9
9	3	1·2
10	1	·8
11	3	1·6
...	323	323

I have not come across a distribution really represented by this type, but I give an unsuccessful attempt to apply it to a series of withdrawals. The constants were

β_1	4·97
m	29·69
b	57·14
$\log y_0$	54·3563
distance of mean from origin	59·205

In calculating areas we use $y_0 a^{-(m-1)} \div (m-1)$ as the area from a to ∞.

Type XII.

$$y=y_0\left(\frac{\sigma(\sqrt{3+\beta_1}+\sqrt{\beta_1})+x}{\sigma(\sqrt{3+\beta_1}-\sqrt{\beta_1})-x}\right)^{\sqrt{\frac{\beta_1}{3+\beta_1}}}$$

Range from $x=\sigma(\sqrt{3+\beta_1}-\sqrt{\beta_1})$ to $x=-\sigma(\sqrt{3+\beta_1}+\sqrt{\beta_1})$.

The origin is at the mean.

$$y_0 = \frac{N}{b\Gamma(m+1)\Gamma(1-m)}$$

where $m = \sqrt{\frac{\beta_1}{3+\beta_1}}$ and $b = 2\sigma\sqrt{3+\beta_1}$.

When μ_3 is positive the negative sign is taken for the square roots.

The limit of the curve when $\beta_1 = 0$ is a horizontal line.

The criterion is $5\beta_2 - 6\beta_1 - 9 = 0$.

Example.

Frequency	Graduation	Ordinates
...	2	18·5
33	31	31·6
...	...	40·6
53	49	49·3
...	...	57·0
65	65	65·2
...	...	73·4
81	83	82·4
...	...	92·3
101	103	103·3
...	...	114·6
131	134	132·9
...	...	155·5
186	191	186·0
...	...	244·2
350	342	405·4
1,000	1,000	...

The mean is ·051 after the centre of the 131 group. The constants are

μ_2	4·266
μ_3	−7·688
μ_4	48·154
β_1	·761
β_2	2·646
$5\beta_2 - 6\beta_1 - 9$	·368

$$y = 87{\cdot}2\{(5{\cdot}808 + x) \div (2{\cdot}204 - x)\}^{\cdot 45}$$

In addition to the graduation a number of equidistant

ordinates is given. They show that the curve rises abruptly, then less abruptly and then again more abruptly. The withdrawals in select tables are sometimes of this shape (*e.g.*, Japanese experience age 52, females). A somewhat similar twist occurs in a population curve.

U-shaped Curve.

This shape arises in Type I when m_1 and m_2 (or m in Type II) are negative. There are difficulties in fitting it to statistics because we do not know how to adjust the rough moments. The figures given in the table of examples (p. 202, col. 6) were found by calculating the areas of the curve

$$y=10\left(1+\frac{x}{8}\right)^{-\cdot 7}\left(1-\frac{x}{4}\right)^{-\cdot 35}$$

The limit of the U-curve is two separate blocks of frequency at the ends of the range. This limit is reached when

$$\beta_2-\beta_1-1=0.$$

The curves with which we have dealt are rare and in practical curve-fitting may be avoided, for they depend on certain definite values of β_1 and β_2, and the chance of reaching these exact values is negligible. In other words, if the object of fitting a curve in any particular case is to obtain the closest agreement between the actual figures given and the graduated figures, then Types I, IV and VI are all that are necessary, for the other types being transition types and depending on specific values of β_1 and β_2 need not arise. If, however, our object is to study probability in a wider sense, the transition types are of importance and they may, of course, be properly used when the values of the β's only differ from those indicated by the criteria to a small extent. This "small extent" means within the limit suggested by the probable errors of the β's.

I may here deal with a little difficulty that students sometimes encounter in connection with the Types I and VI which can be expressed in the same algebraic form, and may more obviously be found with Types VIII, IX and XI, all of which can be written in the form hx^k, and the question may be asked why we should not fit hx^k from a to b and find h, k, a and b from the equations

for the moments. The answer is that the criteria afford in effect a simplification of the equations and automatically tell us a good deal about the value of the constants and the range of the curve.

The curves arising from the equation

$$\frac{d \log y}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2}$$

give us a means of graduating frequency distributions that vary from two separated lumps of frequency, through U-shaped curves to J-shaped curves, then to the "cocked hat" shaped with which we are familiar; incidentally they give straight lines, the exponential and the Gaussian curve.

In the paper dealing with the curves of which we have given examples Pearson indicates that the whole range of these shapes can be obtained from samples from a large population in which each individual carries any number of characteristics which are correlated together. If the correlation be calculated from m samples of equal size we shall not reach the correlation r known to exist in the whole population, but shall have a frequency distribution giving the number of samples that show various values of the correlation between -1 and $+1$. The distributions will vary with the size of the sample and the value of r in the whole population and these distributions, although they do not usually lend themselves to expression by simple functions, give the various shapes that are found from the frequency curve system we have discussed.

II.—CALCULATION OF THE COEFFICIENT OF CORRELATION IN A TWO-ROW TABLE.

The method of calculating the coefficient of correlation when both the variates are continuous and quantitative is dealt with in Chapter VI of "Frequency Curves and Correlation" and the method to be used when the variates are not quantitatively measurable is given in Chapter VII. There are, however, intermediate cases where one variate is, and the other is not, quantitatively measurable, as, for instance, in the following table relating to the effect of enlarged glands on the weight of children (boys).*

* The method is given by Prof. Pearson in *Biometrika*, Vol. VII, pp. 96 *et seq.*, and the example is taken from that paper.

Weight	Boys with good glands	Boys with bad glands	Total
14	2	...	2
16	3	5	8
18	15	26	41
20	20	40	60
22	28	47	75
24	34	30	64
26	30	31	61
28	29	20	49
30	30	30	60
32	21	14	35
34	18	11	29
36	18	5	23
38	6	7	13
40	5	2	7
42	7	3	10
44	1	...	1
46	...	...	...
48	3	...	3
50	1	...	1
52	1	...	1
...	...	...	...
62	2	...	2
Total	274	271	545

If the reader considers a volume of frequency built out of a complete table such as that for endowment assurances on p. 107 of "Frequency Curves and Correlation," or out of a correlation table giving relative ages of husbands and wives, he will see that he has a complete distribution. Now if a volume of frequency be cut off from such a complete volume by a vertical plane at a given value of one variate, then the vertical through the centroid of this volume cuts the regression line. The vertical plane in the two-row table is at the division of the rows; in our example where the good glands end and the bad ones begin. If $\bar{p}$ and $\bar{q}$ be the co-ordinates of the point of section where the vertical through the centroid of the volume cut off cuts the regression line, then we have, σ_1 and σ_2 being the standard deviations of the two variates and r the correlation

$$\bar{p} = r\frac{\sigma_1}{\sigma_2}\bar{q}$$

or

$$r = \frac{\bar{p}}{\sigma_1} \div \frac{\bar{q}}{\sigma_2}$$

Now $\bar{p}$ is the mean value of the quantitatively measurable

variate for all the pairs with a certain one of the alternative variates, in our example, the mean weight of boys with bad glands and σ_1 is the standard deviation of all the boys. We cannot calculate $\bar{q}$ and σ_2 in a similar way because they relate to glands of which no quantitative measure is available. If we assume the non-measurable variate (glands) to follow the normal probability distribution the proportion of the non-measurable variate gives, with the help of tables of the probability integral, the ratio of $\frac{y}{\sigma_2}$ for the distance from the mean at which the division of this variate occurs, and then

$$\frac{\bar{q}}{\sigma_2} = \frac{N}{\sqrt{2\pi\sigma_2^2}}\int_y^\infty ye^{-\frac{1}{2}\frac{y^2}{\sigma_2^2}}dy \div \frac{N}{\sqrt{2\pi\sigma_2}}\int_y^\infty e^{-\frac{1}{2}\frac{y^2}{\sigma_2^2}}dy$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y/\sigma_2)^2} \div \frac{1}{\sqrt{2\pi}}\int_{y/\sigma_2}^\infty e^{-\frac{1}{2}y^2}dy$$

The numerator is z and the denominator $\frac{1}{2}(1-a)$ in the notation used in Sheppard's tables of the probability integral.

The working of the numerical example may help to make the method clearer; it is as follows:

The mean weight of all the boys is	27·7522
The standard deviation is	6·7502
The mean weight of boys with bad glands is ...	27·3737

$\frac{1}{2}(1-a) = 271 \div 545 = \cdot 4972$

$\frac{1}{2}(1+a) = \cdot 5028$ and this value corresponds with

$z = \cdot 3989$ in Sheppard's tables.

The correlation of good glands and weight is

$$\frac{(27\cdot 7522 - 27\cdot 3737)}{6\cdot 7502} \div \frac{\cdot 3989}{\cdot 4972}$$

$$= \cdot 070$$

III.—THE CORRELATION RATIO.

The correlation coefficient which is discussed in Chapters VI and VII of "Frequency Curves and Correlation" is a satisfactory measure of correlation when the regression line is a

straight line, that is when the means of the columns (and the means of the rows) are in a straight line; but in other circumstances its use is open to objection. In Chapter X another method was described which is not open to the same objection, and it may now be of interest to add a short note on a function known as the "correlation ratio" (η) which is a useful measure in many cases.

The value of η is given by

$$\eta^2 = \frac{S\{n_x(\bar{y} - \bar{y}_x)^2\}}{N\sigma_y^2}$$

where $\bar{y}_x$ is the mean of the y's corresponding to the particular array x, n_x is the number of cases in the array x and $\bar{y}$ is the mean of all the y's. The summation extends over all the arrays. In a similar way we can work from the y-arrays and have

$$\eta^2 = \frac{S\{n_y(\bar{x} - \bar{x}_y)^2\}}{N\sigma_x^2}$$

These values of η will not be the same except in the limiting case when regression is linear. The correlation ratio can alternatively be expressed as the ratio of the standard deviation of the means of the y-arrays, each array being weighted with the number in it, to the standard deviation of the y's. The values of η vary between 0 and 1 and are equal to those of r when regression is linear.

Taking the example on p. 107 of "Frequency Curves and Correlation" we should find η as follows:

Mean unexpired term in each column y_x	Deduct mean of whole (20·312) $y_x - y$	$(y_x - y)^2$	n_x	$n_x(y_x - \bar{y})^2$
10·333	−9·979	99·6	6	598
13·250	7·062	49·9	4	200
13·176	7·136	50·9	17	865
16·113	4·199	17·6	62	1,091
17·230	3·082	9·50	584	5,548
20·141	·171	·029	643	19
21·877	+1·565	2·45	1,098	2,690
21·665	1·353	1·83	388	710
21·500	1·188	1·42	60	85
27·625	7·313	53·4	8	427
...	...	...	2,870	12,233

$$\therefore \qquad \eta^2 = \frac{12233}{2870 \times (7{\cdot}6067)^2}$$

$$\text{or} \qquad \eta = {\cdot}271$$

The figure 7·6067 is the value of σ_2 on p. 118, multiplied by 5 the unit of grouping. The probable error is approximately ·012 being calculated as $\cdot 67449 \frac{1-\eta^2}{\sqrt{N}}$, which is an accurate expression when regression is linear and can be used as a guide in cases which do not depart widely from this condition.

Working similarly with the maturity ages we obtain the following :

$x_{\prime\prime} - x$	$(x_{\prime\prime} - x)^2$	$n_x(x_{\prime\prime} - x)^2$
3·48	12·11	678
2·20	4·84	832
1·38	1·90	821
·64	·41	273
·35	·12	82
·65	·42	227
2·71	7·34	1,813
3·81	14·52	1,117
5·27	27·77	222
7·77	60·37	60
...	...	6,125

$$\therefore \qquad \eta^2 = \frac{6125}{2870 \times (5{\cdot}6818)^2}$$

$$\text{or} \qquad \eta = {\cdot}257 \pm {\cdot}012$$

It may be of interest to set out the various values found as measures of the correlation of the same table :

Correlation coefficient	·254
Correlation ratio (first value)	·271
Correlation ratio (second value)	·257
Contingency method (Chapter X)	·287

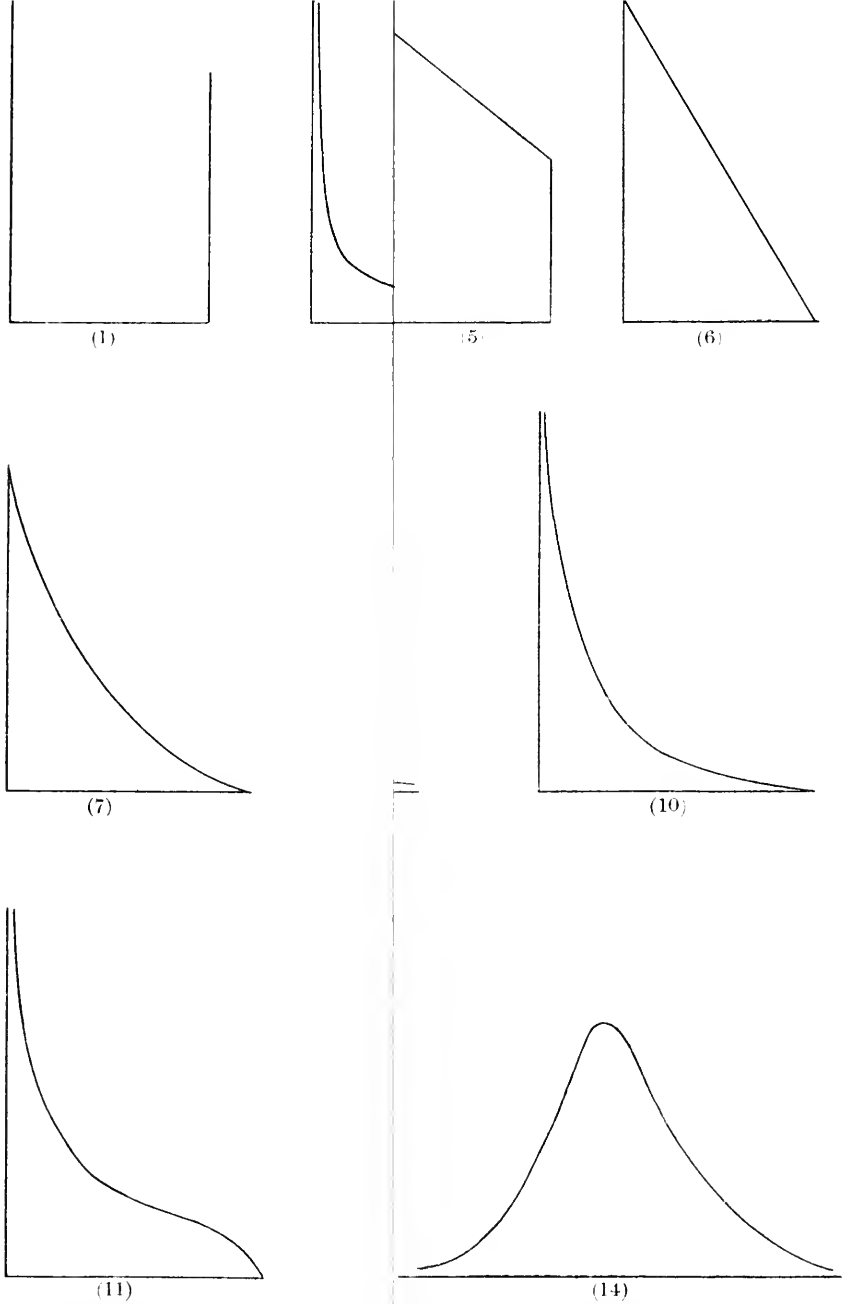
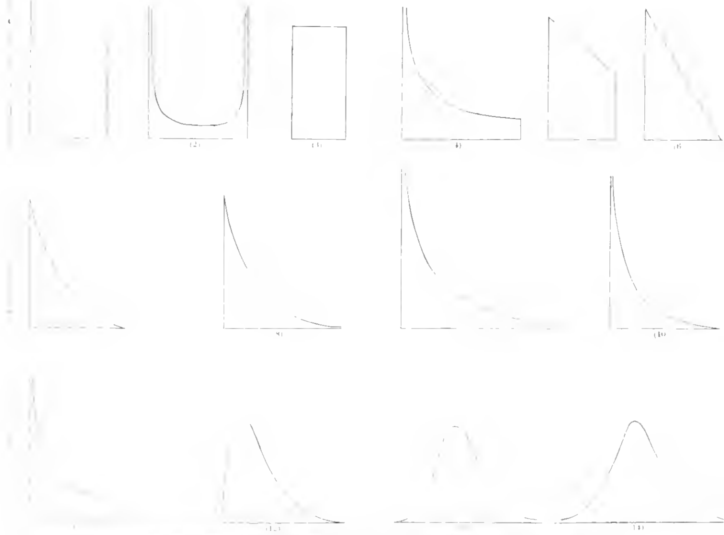


Diagram showing transition from *mmon "cocked hat" shapes.*

(1) *represents two separated blocks* *of* (3) *is, as it were, the bottom piece of the* U-*curve and the Type* VIII *curve* *ached.* (7) *is Type* IX *and* (8) *is the exponential. The next two curves* (9) *a* *proceed to Types* I, III, IV, V *and* VI, *curves of the "cocked hat" shape : th*



... Type ... passes through U-shaped and J-shaped ... to the ... cocked hat shape.
... into the U-shaped ... in (2). The horizontal straight line ... (3) is, as it were, the bottom piece of ... VIII ... U curve. (5) and (6) ... straight line are ... (7) is Type IX and (8) is the ... (9) and (10) ... J-long ... of Types III and I and (11) ... Type XII. ... to Types I, III, IV, V and VI, ... in (12), (13) and (14).

ERRATA, &c.

Page	
18, line 4 from top ...	For $\Sigma y_r X_r$ read $\Sigma y_r X_r^2$
26, line 5 from bottom ...	For y_n read y_{n-1}
27, Table V, total, and 28, line 3 from top ...	For 208·38 read 207·38
30, line 8 from top ...	After IV add "of Table I"
30, line 9 from top ...	After "contact", delete remainder of sentence, and substitute "and in "I the rough moment should be "used. In II and V there is more "doubt, and in the calculation of "the moments for II (*see* p. 55) no "adjustment was made."
33, heading of third column of table	For p. 24 read p. 27
37, line 10 from bottom ...	For n read $n+1$
39, line 15 from top ...	Delete $y=0$
40, line 16	Insert a *minus* sign before the right-hand side of the equation
43, line 12 from bottom ...	Insert b_2 as a multiplier for the right-hand side and as a multiplier for the indices in the following line
43, line 4 from bottom ...	The suffix 2 has been dropped in some copies in the index to the second bracket
44, line 6 from bottom ...	Insert $\frac{1}{b_2}$ before second integral
44, lines 4 & 5 from bottom ...	For $\frac{c}{A}$ read $\frac{c}{Ab_2}$
44, line 4 from bottom ...	For $2b_2$ read $\frac{1}{2b_2}$
45, line 9 from top ...	The sign after the first bracket should be *minus*, and consequently the sign of the corresponding index in the two following lines should also be negative
45, line 10 from top ...	For b read b_2; and for $\frac{b_1}{b_2}$ read $\frac{b_1}{2b_2}$
52	The fourth moment $\nu_1 = \nu'_4 - 4d\nu_3 - 6d^2\nu_2 - d^4$
59, line 3 from top ...	In some copies the suffix, 1, to the lower limit of the integral and the *minus* sign before the limit have been dropped

PAGE

59, lines 4, 5 & 12 from top. For $z - 1$ read $1 - z$

60, line 5 from top ... For $-2r^2$ read $+2r^2$

60, line 10 from bottom... The denominator should be $3\beta_1 - 2\beta_2 + 6$

63, heading to col. (5) ... For y_0 read $\log y_0$

72, heading to col. (7) ... Insert *minus* sign before m

74, line 9 from bottom ... For $\phi + \frac{1}{2}\pi = \theta$ read $\frac{\pi}{2} = \theta + \phi$

74, line 2 from bottom ... For $\cos^{r+n+1}\theta$ read $\cos^{r-n+1}\theta$

76, line 7 from top ... For $\frac{8r}{z}$ read $\frac{8r^2}{z}$

83 ... ... Instead of lines 5 and 6 read

q_2 and $-q_1$ are given by

$$\frac{1}{2}\left\{r - 2 \pm r(r+2)\sqrt{\frac{\beta_1}{\beta_1(r+2)^2 + 16(r+1)}}\right\}$$

83, last line and line 3 from bottom ... For $\frac{r+2}{r+2}$ read $\frac{r+2}{r-2}$

85, heading to col. (5) ... The suffix, 2, has been dropped in some copies

89, heading to fourth col. of table ... For a read $\frac{1}{2}(1+a)$

99, line 2 from top ... For y read y_0

99, insert as a note to the formulæ in § 12:

"Care is necessary with regard to the value used for y_0, and consequently with regard to A and B. If Sheppard's tables of ordinates (z) be multiplied by, say, 10^5 and used as the exposed to risk, the values of A and B resulting from the work will be $N_1 \div (10^5\sigma)$ and $N_2 \div (10^5 H\sigma)$. The reason is that his tables are in terms of standard deviation. In the example, pp. 99, 100, § 14, this is obscured because the assumed standard deviation is 10."

99, line 15 from bottom... For "duration" read "deviation"

102, line 10 from top ... For 29·268 read 25·093

103, line 12 from top ... For 1·0875 read 1·1048

111, lines 4 & 6 from bottom and foot-note ... $\frac{1}{2\sigma^2}$ should be $\frac{1}{\sigma^2}$, and the suffixes of the σ's should be interchanged

112, top line ... ... For $r = -\frac{c_{12}}{c_1c_2}$ read $r = -\frac{c_{12}}{\sqrt{c_1c_2}}$

112, line 3... ... ... For 2π read π

PAGE

118, second line below table should read

Mean unexpired term $= 22{\cdot}5 - 1{\cdot}68815 = 19{\cdot}81185$

123, line 6 from top ... For σ_2 read σ_3

123, line 7 from top ... For $\frac{\sigma_2}{m_2}$ read $\frac{\sigma_3}{m_3}$

152, line 2 from bottom ... Insert x^{m-1} under integral

152, line 1 from bottom ... For x^{m+1} read x^{m-1}

153, line 2 from top ... The limits of the integral on the left-hand side are 0 to ∞, and the expression under the integral on the right-hand side should be $e^{-y}y^{m+n-1}$

153, line 5 from bottom should be $\Gamma(x+1) = \sqrt{2\pi x}\, x^x e^{-x} e^{\frac{1}{12x}}$

153, line 3 from bottom should be

$$\log_{10}\Gamma(x+1) = \log_{10}\sqrt{2\pi} + \left(x + \frac{1}{2}\right)\log_{10}x - \left(x - \frac{1}{12x}\right)\log_{10}e$$

157, line 13 from top should be $-(x^2 + y^2 - 2rxy)r(1 - r^2)^{-2}$ &c.

158, footnote ... ... The last symbol should be r_n not r^n

165... ... ... The volume of tables for statisticians has been recently published by the Cambridge University Press.

National Insurance Acts. The Departmental Committee's further and final Reports.

THE further report of the Departmental Committee on Approved Societies Finance and Administration [Cd. 8396] deals with the Committee's recommendations on the second part of their reference, namely: " . . . to consider how far the "work of Approved Societies could be simplified and its cost "reduced, without detriment to the interests of insured persons, "by amendment of the Acts and Regulations."

In this report Sir Gerald Ryan and his colleagues devote themselves for the most part to a critical survey of the Acts in their actual operation. Many problems arose in the adaptation of a scheme of compulsory insurance with a common financial basis to the varying needs of different classes of the industrial community, and within the limits which the general principles imposed, various solutions were found and incorporated in the

Statutes. The basis of some of these solutions may be traced in the several actuarial reports and other official publications which were issued during the course of the Parliamentary proceedings of the Bill of 1911, but in respect of others there are no ready means of ascertaining the principles on which the questions involved were ultimately settled; though much information can, of course, be obtained by those who have the patience and industry to search the official reports of the Parliamentary proceedings of that strenuous session. The Committee has thus found ready to hand the opportunity of doing a valuable piece of work in the examination and analysis of the basic principles of the minor (but, nevertheless, important) developments of the Acts in various directions and their "further report", extending to nearly 90 pages, is consequently a more elaborate production than usually results from the deliberations of departmental committees. To the student of this phase of social economics the report is of special interest in that it explains and often justifies the basis of many provisions of the Act whose *rationale* is not self-evident, and examines and discriminates between the good and the bad in many proposals which have been advanced for the improvement of the national scheme.

It follows that many of the recommendations are concerned with simplification of the administrative machinery of the Acts, and while these are of great importance and interest to those engaged in the conduct of Approved Societies' affairs, they do not involve actuarial considerations. We propose, therefore, to deal mainly with the recommendations which involve the financial side of the Acts.

In proceeding with Part I of their reference, the Committee thought it sufficient to rely upon the statistical information accumulated by the Insurance Commissioners during the period for which the scheme of the Act had been in force, but in dealing with questions of simplification and economy, the Committee naturally sought the assistance of those who had first-hand knowledge of the details of Approved Societies' administration. Invitations were accordingly issued to officials of Approved Societies and other interested persons to submit evidence as to any difficulties and undue cost incurred in working the National Scheme, and as to the remedies which, in their opinion, would remove or reduce them. The Committee had the advantage of hearing evidence from sixty representative witnesses and of

considering memoranda presented by many Approved Societies and other bodies which were specially qualified to speak upon the subjects under enquiry.

The Committee state that the general impression created by the evidence is two-fold; first, that fundamental reconstruction is needed in regard to a small but important group of questions and, secondly, that a reduction is desirable in the number of special classes for which differential contributions or benefits were provided. They have been impressed by the opinion, very strongly and widely held, that the actual demand for exceptional treatment is much more limited than was anticipated, and that the labour and expense entailed in the maintenance of those subdivisions of the insured population would be far greater than any advantage which could thereby accrue.

They cannot concur in the suggestion put forward by some witnesses that all special classes should be swept away. They are convinced that a National Scheme of Compulsory Insurance must contain special provisions to meet the needs of certain minority classes (*e.g.*, seamen serving in foreign going ships) to which the ordinary scheme would not be appropriate.

In order, however, to reduce the number of special classes, they recommend that alterations should be made in the provisions of the Acts affecting the following classes:

(1) Voluntary contributors.
(2) Late entrants, minors and persons ceasing to be insured.
(3) Aliens.
(4) Persons whose employers are liable to pay wages during sickness.
(5) Low wage earners.

As regards voluntary contributors (who, it is curious to learn, number only some 28,000 out of the 800,000 originally estimated to be eligible for this class of insurance) it is recommended that the class be closed to new entrants into voluntary insurance. It is considered, however, that the right given by the Act of 1911 to persons who have been employed contributors to continue in insurance as voluntary contributors on giving up insurable employment, should be preserved in those cases where contributions have been paid for a sufficiently long period to create a substantial insurance interest. It is recommended that this qualifying period of employment should be two years.

In the case of those existing voluntary contributors (about

4,000 in all) who at present pay contributions at rates depending on age and in excess of the "flat" rate, it is recommended that their contributions should be reduced to that rate and their Societies compensated for the reduction by the grant of reserve values.

Under the existing provisions of the Acts, a person entering insurance after the expiration of the first 65 weeks of the Acts, is subject permanently to a reduced rate of sickness benefit, unless he pays the additional contribution or lump sum necessary to secure him full benefits. A reduced rate of benefit is also payable for the time being in the case of an unmarried minor unless he can satisfy his Society that one or more of his family are wholly or mainly dependent upon him.

The above provisions have proved in practice to be very difficult to administer, and have unreasonably complicated the accounts and registers of Societies. It is accordingly proposed that late entrants and minors should be treated on identical lines, the rate of sickness benefit being reduced during the eighteen months following the ordinary waiting period of six months, to 6*s*. a week for men and 5*s*. a week for women. The result would be that in the normal case of a person entering into insurance at the age of 16, the sickness benefit would be payable at a reduced rate until age 18 and after that would be at the full rate.

The question of the benefits to late entrants is closely allied with the conditions governing lapse from and re-entry into insurance. The provisions of the Acts in this respect are somewhat obscure, and it is understood that the advice given to Societies by the Insurance Commissioners, namely, that a person who goes out of compulsory insurance by a change in his occupational condition remains nevertheless a member of his approved society, has been based upon a balance of general considerations recognizing, however, that further legislation would be necessary to clear up the position.

After a general review of the situation, the Committee recommend that persons who cease to be employed within the meaning of the Act should continue to be entitled to the normal benefits for one year after so ceasing. They recommend, further, that membership of the Society should terminate when insurance ceases.

The corollary to these proposals is that re-entrants into insurance should be placed on the same basis with regard to

benefits as persons who are entering insurance for the first time. Where, however, a person re-enters insurance within two years of the date at which he ceased to be a member of his Society, it is recommended that he should be entitled to claim re-admission to the same society.

As regards financial provisions, it is recommended that the transfer values of all persons who go definitely out of insurance (reduced to meet the liability which the Society has incurred by continuing benefits for a year after contributions had ceased) be carried to a Central Fund, and that out of this fund be provided the reserve values required for late entrants or re-entrants into insurance, which at present (so far as required) have to be provided by the general sinking fund. The result would, of course, be that any sums arising from lapse profits would enure for the benefit of insured persons as a whole and not for the benefit of particular societies.

The provisions of the Acts with regard to aliens (Section 45) and cases where the employer is liable to pay wages during sickness (Section 47) are found in practice to apply to only a small body of persons, while imposing upon societies great accounting complexities. It is accordingly recommended that these special provisions be abolished and that these classes be treated for insurance purposes on the normal lines.

In regard to Section 47, it is suggested that any loss (due to low average age) which may emerge at the first valuation in respect of members who have contributed at the special rate, should be earmarked and made good from the Central Fund referred to above.

In the case of certain other special classes, recommendations are made which lead to simplification in working.

One of the most difficult sections of the Act to administer is that dealing with the special provisions in respect of married women and the complicated options which arise when a woman marries, and again when a married woman becomes a widow.

Those who are familiar with the administrative work arising out of the above features of women's insurance will agree that one of the greatest difficulties is to ascertain the real intention in regard to the continuance in insurance of a woman who marries. Where the circumstances are such as to raise the presumption that benefit will be claimed a few months after cessation of employment, it is almost impossible to apply the Section in more than form. The society has to rely largely on the genuineness

of a statement as to her intention to return to work after recovery, and from the nature of the case no test of a statement made in these cases is possible. On the other hand, the important case of *Davidson* v. *New Tabernacle Approved Society*, which goes far to fetter the discretion of a society which is essential to the successful working of the section, renders the position of insured women very unsatisfactory and emphasises the need for early legislation in the matter.

In order to meet the above difficulties and to simplify the administrative work in other respects, the Committee recommend the extension to every woman on marriage, whether she intends to continue in employment after marriage or not, of the permission to select one of a small number of options which at present is confined to women who have given up employment. Any woman who, in the exercise of her option, elected to terminate her existing insurance but continued as, or subsequently became, an employed contributor, would be treated as a new entrant with fresh waiting periods for benefits.

Under the scheme recommended by the Committee every employed woman on marriage would have the option of taking (subject to certain qualifying conditions and to arrears):

(1) A marriage benefit of £2 (payable on notice of marriage) together with the right to medical and sanatorium benefits for one year; or

(2) A new voluntary insurance for medical and sanatorium benefits, and a maternity benefit of £1 at a contribution of 3d. a week; or

(3) A free insurance for one year for the ordinary benefits of an employed contributor (the position in this case being the same as that of any insured person on cessation of employment).

At the end of the year following marriage, women who had taken the marriage benefit or the free insurance and had not again become employed would cease to be members of their societies. If, and when, they again become employed contributors they would be treated as new entrants into insurance.

Under the provisions of the scheme, payments to the Central Fund would be made by the Society on the marriage of each insured woman in lieu of the present payment to the Married Women's Suspense Account, and reserve values would be provided out of that fund in the case of all women who continued in employment on marriage or subsequently re-entered employ-

ment. While thus avoiding difficult administrative questions as to the circumstances in which new reserve values are claimable this proposal would achieve a further object of considerable importance.

The financial basis of women's insurance provides for the constant insurance of one-seventh of the married women who were insured before marriage; and while the composition as regards individuals of this employed proportion may vary from time to time, some women going out of insurance and others re-entering, a society is not injured financially so long as it has only to provide insurance for one-seventh of those of its members who have married. Under the present system, with its flat rate contribution and uniform scale of reserve values, it is assumed that the same proportion of women will continue in employment after marriage in the case of every society. It is, however, notoriously the fact that the proportion of employed married women varies very widely in different industries, and under existing conditions this variation must tend to affect the financial position of women's societies quite apart from any questions of occupational risk or administrative efficiency.

The new proposal would eliminate this cause of financial disturbance and, coupled with the further provision for the claims of married women which was recommended in the Committee's interim report, should provide an effective steadying influence on the finance of women's insurance.

In connection with these proposals, it is recommended that the present Married Women's Suspense Account be amalgamated with the Central Fund and that for the present separate accounts should be maintained in the Central Fund for men and for women respectively.

Another difficult and complicated section of the Act is that relating to seamen serving on foreign-going ships (Section 48). Under the scheme of that Section, the seaman ceases to be liable for further contributions when he has completed 42 weeks in any year on foreign articles, although the employer's contribution has to be paid for every week of employment. This differentiation has caused many difficulties in practice, particularly as it appears that the proportion of seamen who alternate between the foreign and the home trade is considerable.

The Committee accordingly make various recommendations which simplify the working of the Section and at the same time secure the adjustment of certain inequalities of treatment in

regard to the benefits of seamen which have been found to arise in practice.

In other sections of the report the Committee make recommendations on various matters which lead to simplification of working and improvements in procedure. The outstanding proposal under these heads is that of a modification of the scheme in regard to arrears of contributions.

The proper method of treating arrears in an insurance scheme which depends on contributions compulsorily linked up with employment is an intricate but extremely interesting question. Up to a certain point the contingency of unemployment and consequent loss of contributions can be covered by the premium, and some approved society administrators have advocated the application of this principle (which is already recognized by the Acts) to its extreme extent. It is, however, only in theory that the benefits insured to the individual can be divorced completely from the number of contributions which he and his employer pay. To what the extreme application of the doctrine would lead is clearly shown by the Committee.

On the other hand the Committee find some ground for dissatisfaction with the principle of the Act of 1911 which seems to preserve to the individual any advantage that may accrue from his own abundance of employment. They advocate the freer use, for mutual help, of any profits which may result from the excess contributions of the fully employed and they propose a scheme designed to secure this end and accordingly to limit the loss of benefits (or the equivalent money charge) to which the under-employed must submit. They take the opportunity thus afforded to simplify the general provisions as to arrears and to promote arrangements which shall harmonize with the broad plan of the Act under which the smaller financial questions are left (as in a national scheme must always be the case) to the welding operation of the general average.

Not the least interesting part of the Committee's work on this subject is the examination of the fundamental difference between compulsory and voluntary insurance. Many administrators of approved societies appear to have advocated the treatment of arrears in State insurance on the plan with which the friendly societies are familiar, and under which arrears accumulate until an opportunity arises for deducting them from benefits or until their amount involves "exclusion" or "lapse." For the first time, so far as we are aware, it is made clear in this report

why such reasonable incidents of voluntary insurance are inappropriate to a compulsory system.

In their final report (Cd. 8451) the Committee examine subjects involving social questions which embrace a wider field than that of Approved Societies finance and administration, although by their terms of reference they are limited to a consideration of the effect of these questions in relation to the working of Approved Societies.

The subjects examined are three in number, namely:

(1) Section 63 of the Act of 1911 relating to enquiries into causes of excessive sickness, &c.

(2) The suggested payment of sickness and disablement benefits in sickness due to venereal disease.

(3) The suggested payment of such benefits in the case of tubercular disease during the period of after care—that is, after an insured person has ceased to be qualified for the ordinary sanatorium treatment and has in some measure regained the power to work.

Under the first of these subjects, the Committee draw attention to the fact that a flat rate basis as a criterion of excessive sickness is unsuitable, and they recommend that in place of employing a uniform standard of sickness for the purpose of measuring a loss in connection with a claim against employers or others responsible for preventably bad conditions, the loss sustained by a society should be assessed on the basis of the actual facts of the case, so as to permit of variations which would take account of the normal sickness of the members concerned.

In the altered circumstances, the Committee suggest that the right of a society to take action under the Section should be subject to their satisfying the Insurance Commission that a *primâ facie* case has been made out.

The second heading refers to a subject which is giving rise to considerable discussion at the present time. The present practice of Approved Societies is to withhold payment of benefit in cases of incapacity due to misconduct, and it has been strongly urged by responsible medical opinion that this practice should be abrogated on grounds of public policy in the case of sickness arising from venereal disease. It is argued that by paying benefit in the early stages of venereal disease, societies will provide the means for insured persons to submit themselves to proper treatment and thus to avoid much incapacity in later life which

would involve the payment of sickness or disablement benefit for prolonged periods.

While the Committee feel that it cannot be taken as established that such a change in procedure would not affect the finances of societies adversely, yet on consideration of the various claims and contentions which have been placed before them, they strongly recommend its adoption by societies. They suggest also that societies might consider the desirability of applying any alteration of practice in this respect upon which they may decide, to their operations independent of the Act as well as to their corresponding operations under it. It is pointed out that societies which are concerned to maintain a high standard of personal conduct among their members, can achieve this end by retaining any powers of expulsion in cases of wilful misconduct, which they may possess under their rules.

In connection with the question of the payment of benefit to tuberculous patients during the period of after-care, two interesting memoranda are submitted, one by Dr. P. C. Varrier Jones, Acting Tuberculosis Officer for the County of Cambridge, and the other by Mr. George Fraser, Chairman, and Mr. W. E. Whyte, Clerk, of the District Committee of the Middle Ward of the County of Lanark, with reference to the Hairmyres Colony Scheme.

It would seem that medical opinion definitely inclines to the view that an essential part of the treatment of tubercular disease should consist of a course of graduated exercise, increasing in duration and intensity as the strength of the patient is built up. The witnesses represent that the qualifying condition of incapacity for work, strictly applied, is unsuitable in the case of tuberculous patients and urge that Approved Societies should pay benefit to such patients when convalescent for periods of partial incapacity. It is also considered in many quarters to be exceedingly desirable that, in order to expedite the patient's recovery, he should be paid a wage as soon as, and to the extent to which, he performs remunerative work.

The Committee think that it may fairly be held that so long as an insured person is resident in an Institution approved under the Act, and is undergoing treatment under proper control and supervision, the fact that he is required as part of his treatment to perform physical exercises, which may differ from work only in that they are carried out under close medical supervision, should not affect his claim to receive, and his Society's liability

to pay, sickness or disablement benefit. If under the rules of the institution, payment should be made for "work" so performed, that fact might, they think, be properly disregarded, provided always that it is clear that the "work" is a mere incident of treatment.

They cannot, however, see their way to advocate an arrangement whereby payment would be made in cases of partial incapacity where the patient has returned to his own home and is engaged in some work. The difficulty of securing uniformity of certification in such circumstances would be enormous, and it is well known that the system of payment of benefit as of right to persons who are not wholly incapacitated from work has, in the history of friendly societies, proved to be disastrous.

Continuous Valuation Machinery. By P. H. McCormack, A.I.A.

ALTHOUGH several important papers have appeared in the *Journal* during recent years on the subject of valuation methods, the closely-allied question of the tabulation of valuation data has scarcely been touched upon for some time. The following note may therefore be of interest; it has been prepared after an examination of several systems now in use, and contains a description of the machinery recently adopted by an office which had not hitherto employed the continuous method of classification.

Actuaries have long recognized the advantage of the continuous method, whereby the statistical particulars are, as it were, built up from year to year, over the old system of taking them out afresh at each valuation. Most offices have therefore adopted the continuous principle in one form or another, but nearly every office has solved the problem in its own particular way, with the result that several different practical applications of the system are now in use. Probably the most important difference is to be found in the methods of tabulating the policies in force, some offices employing both class-books and cards for this purpose, others using either books or cards. The double record necessarily involves a duplication of labour, and does not appear to have any advantages which cannot be obtained by a simpler system, while the class-book method alone possesses none of the elasticity of the card system, with its facility for sorting and resorting in any order and its total exclusion of all data which are no longer required. The question has lately

received consideration in several offices, and where a change has been made, the tendency has been in the direction of abandoning valuation class-books in favour of the card system alone. The latter system forms the basis of the machinery explained in this note. The work is described in full from the commencement, although some slight modification would be required in the event of the previous existence of class-books, which would no doubt be useful in the initiation of the new system, if it were desired to change.

The main object of the system is to enable the policy particulars in each classification group as at 31 December in every year to be accurately ascertained at the earliest possible date, thus providing the necessary basis for an annual valuation in a form which will enable the statutory returns to the Board of Trade to be prepared. To achieve this object it is necessary:

1. To tabulate the data in the simplest form consistent with the Assurance Companies Act.
2. To choose a method which makes it convenient to reconcile the particulars of policies in force at any 31 December with those of a year earlier. These figures will be referred to as the "totals in force" for brevity.
3. To make the various elements leading up to this balance, *e.g.*, new policies, claims, surrenders, &c., self-balancing, so that errors may be easily traced.
4. To avoid duplication which does not serve some purpose.
5. To make the method so straightforward that it can be used and understood by clerks having no actuarial training.

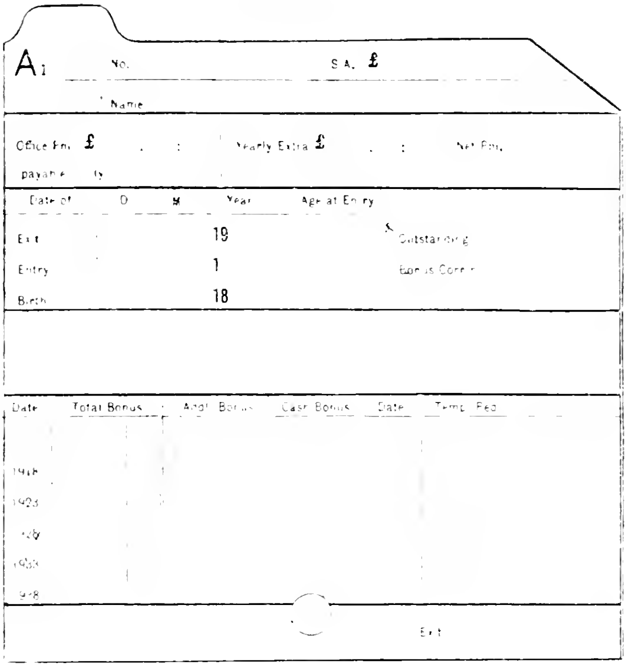
It is also desirable to arrange the valuation data so as to provide information required for general office purposes, such as the bonus record, particulars of the last premiums in limited payment and endowment assurance cases, and the expiry of term policies.

Method of Classifying the Existing Policies.

In the first place cards were written from the policy registers for all policies in force. The particulars having been checked, bonuses were inserted from the bonus books and the necessary actuarial functions, &c., calculated and checked. The cards were then put away until all the existing policies had been dealt with.

The following is the form of card used for whole life policies, very slight modification being required to adapt it to other classes of assurance:

Dimensions 5″ × 5″.



The continuous valuation system must be started from some fixed date, and it was found convenient to arrange a future date so that the preliminary work could be then completed; a little care was necessary to see that alterations in the meantime were carried through, but to ensure accuracy the cards were again read over.

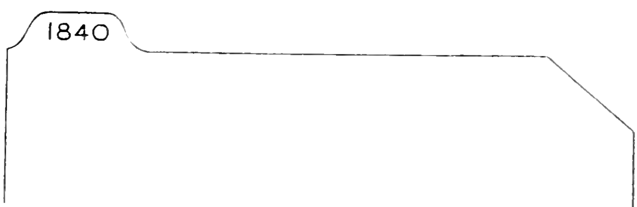
The next stage was to sort the cards to class, for the purpose of tabulating the figures in each valuation group. This process was greatly facilitated by adopting a system of tabs. The cards relating to each of the main classes of policies are not only of distinctive colour, but contain tab projections on the top edge in such a position as to indicate the class of policy, and bear on the tab a note of the valuation group. In addition to convenience in sorting, the system has another important use as a record of

* This item is included because the fractional premiums charged by the office are in all cases instalment premiums.

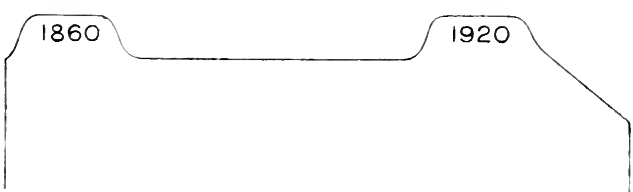
the changes which arise automatically out of the contract in certain classes of policies. It is necessary every year to prepare a list of deferred assurances vesting, last premiums falling due in limited payment cases, half-premium policies becoming subject to the full premium, and so on. For this purpose a special tab is provided recording the date of automatic alteration, and it is, therefore, merely necessary to go through the trays of cards, which are kept in numerical order, and scrutinize those cards whose tabs bear the date of the year for which information is being taken out. The other cards may be turned over very quickly, and the collection of the required details readily effected.

In order to render the process of scrutinizing the tabs as simple as possible, it was decided to have four tab positions on the cards and to use the 3rd and 4th only for dates required in the annual scrutiny, thus reserving the 1st and 2nd positions for group dates only. The positions of the tabs on the cards for the different classes of policies were therefore arranged as follows, the distinction between with and without profit cases being made clear by writing the tab dates in black ink for with profit policies, and in red ink for without profit policies:

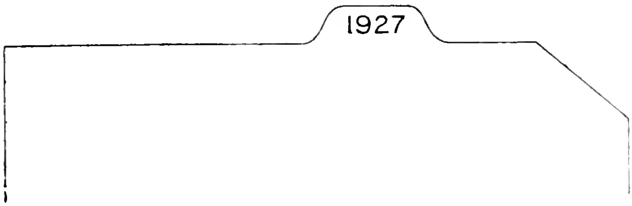
1. *Whole Life Assurances.*—1st tab only, recording the office year of birth.



2. *Limited Payment Whole Life.*—1st and 4th tabs. The 1st tab to contain the office year of birth and the 4th tab the year of last payment.

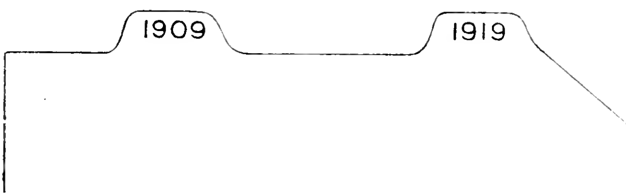


3. *Endowment Assurances.*—3rd tab only, recording the office year of maturity.

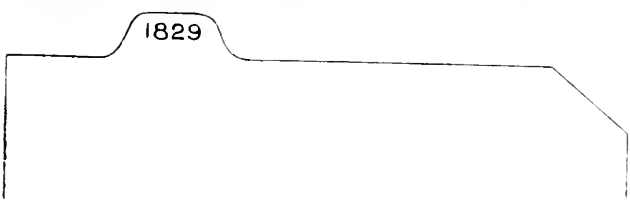


Endowment assurances sometimes mature in the year following that in which they are assumed to mature for valuation purposes, and if an office has such cases as well as assurances maturing on the policy anniversary, the former will be sufficiently indicated by enclosing in square brackets the office year of maturity placed on the tab. In making up for office purposes the list of endowment assurances maturing in 1918, for example, it will thus be necessary to include those cases whose tabs are marked [1917].

4. *Special Class Policies.*—2nd and 4th tabs. The 2nd tab to contain the valuation group date (as, for example, the year of issue in deferred assurances with return) and the 4th tab to contain the date when an automatic alteration takes place (as, for example, the date of vesting of a deferred assurance).



5. *Annuities.*—2nd tab only, recording the office year of birth.



It is desirable to have a space between the positions of the 2nd and 3rd tabs in order to isolate the two right-hand from the

two left-hand tabs, the former alone being required in the annual scrutiny.

Some cards have thus one date tab, and others two; but not more than two dates are required in any case, because if the policy conditions are such as to involve more than one automatic alteration, only the earlier one need be recorded. When that date arrives, the policy will be transferred to another class, and a fresh card written recording the later date. For example, in the case of a deferred endowment assurance with return, a special class card would be written at the outset containing the year of issue, being the valuation group date, on the 2nd place tab, and the year of vesting on the 4th place tab. On the commencement of the risk the case would be transferred to the endowment assurance class, a fresh card being written with the date of maturity on the 3rd place tab.

As an alternative method, the second tab might be dispensed with and the date of automatic alteration written on the back of the remaining tab. It is also possible to use one uniform type of card provided with several tabs, so that the staff engaged on writing up the cards need not consider which kind of card to select in any particular case. The tabs not required would subsequently be cut off.

Process of Tabulating the Initial Figures.

For the tabulation of the figures in each valuation group at the date of commencement of the new method, it was found convenient to employ one of the modern adding machines which gives automatically a typewritten list of the various items. The policy numbers were listed by the machine as well as the other particulars, so that a complete record of the initial policies was obtained.

The cards relating to each valuation group were scrutinized before being handed to the adding machine operator in order to avoid mistakes in sorting, and to secure perfect accuracy the work was done in duplicate.

The figures thus obtained were next entered in a Summary Book, recording the policy particulars in force year by year for each valuation group in a form of which the following is an example:

Valuation Group......

	No. of Policies	S.A.	Rever-sionary Bonus	Office Premium	Extra Premium	Net Premium
In force on.........						
New policies, 19...						
*Misc. additions .						
*Deductions, 19 ..						
In force on 31 Dec. 19......, &c.						

The valuation cards were then sorted permanently into numerical order, which is on the whole the most convenient arrangement and the least liable to lead to errors in sorting. Moreover, it enables the valuation cards to be used conveniently as a complete bonus record.

The cards should be stored in a fireproof cabinet, and in order to avoid mis-sorting it is advisable to make a rule to the effect that every card should be reversed when replaced in the cabinet, so that the corners protrude above the cut edges of the other cards. The cards so replaced should be checked from time to time and returned to their proper position. This precaution should make it very difficult for a card to be mislaid; but if at any time one cannot be found in its correct place, it will be possible to scrutinize all cards bearing upon the tab the same valuation group date as the lost card. The latter will probably be then discovered.

New Policies.

All policies issued after the fixed date referred to above are regarded as new cases, and are not included in the initial figures. Valuation cards are prepared and checked for the new policies, and at intervals of say once a week they are sorted into classification order and entered in the new policy sheets under the appropriate valuation groups. At the end of the year the totals of these groups, after being reconciled, will be carried to the Summary Book, and added to the previous "totals in force."

In order to distribute the work as far as possible throughout

* Miscellaneous additions and deductions are explained subsequently.

the year, it is desirable to make interim reconciliations with the policy register about once a quarter.

The new policy sheets are not in the nature of class books, as they are merely used as a means of analyzing the new policies for the Summary Book. That is to say, they do not record the present position of a policy, but only the particulars at issue.

Reassurances with other offices are dealt with throughout on the same lines as the assurances of the office, but in accordance with the requirements of the Assurance Companies Act they must be kept separate, and special cards and sheets are therefore necessary.

Cancelments.

Separate office cancelment books are used for Deaths, Maturities, Surrenders, and Lapses (including term policies expiring, &c). These books contain columns for sums assured, bonuses, and office yearly premiums, for reconciliation with the valuation department sheets, and there is also a column to indicate classification group. On a cancelment taking place, it is entered in the appropriate cancelment book, the entries in each book during a particular calendar year being consecutively numbered for convenience of reference. The valuation department uses the office cancelment books for extracting the valuation cards, and marks upon the latter the cause of cancelment, with the year and reference number. When taking out the cards, it should be noticed that the facts entered thereon agree with those in the book.

After a batch of cancelled cards have been collected in this way, they are sorted to group and entered in the appropriate cancelment sheets, the totals at the end of the year being carried to the summary. The cancelment sheets must be reconciled with the office books, and in order to ensure a balance being easily effected it is advisable from time to time to read over the cancelment books with the sheets. The system of reference numbers facilitates this operation, as it enables an item read from the sheets, which contain the reference number, to be traced and verified in the books. The valuation group under which the item has been entered in the sheets should be checked with that indicated in the book by means of the special column provided for the purpose in the cancelment book. As in the case of new policies, it is convenient to make an interim reconciliation for cancelments about once a quarter. The advantage of separate books and sheets for each of the four classes of cancelments is

evident in the balancing work, as discrepancies are localized and errors more easily traced than if all cancelments were dealt with together.

The cancelled cards are, of course, kept separately in respect of each kind of cancelment. They will be found useful if the particulars do not balance at the first attempt.

Miscellaneous Alterations.

The office alterations book contains a large number of items which do not affect the valuation particulars, and the valuation department therefore keeps a separate alterations book, arranged for balancing with the additions and deductions sheets. The left-hand pages of this book are used for deductions and the corresponding right-hand pages for additions. Particulars of each alteration affecting the valuation are entered with a note of the group to which the case belongs, each year's alterations being consecutively numbered. For the purpose of transferring items from the office alterations book to the valuation alterations book small cards in the following form are used :

Dimensions $4\frac{1}{2}'' \times 3\frac{1}{2}''$

Class Group	Policy No. Alteration No.	
	Before	After
S.A.		
Bonus ...		
Office Premium		

These cards are also found useful for taking out and altering the main valuation cards, for writing up the miscellaneous

additions and deductions classification sheets, and for facilitating the balance of the books and sheets.

The additions and deductions sheets are of the same pattern as the cancelments sheets. Miscellaneous additions for each valuation group are recorded on one side of a sheet, and miscellaneous deductions for the same group are entered on the back of the same sheet. For example, if a policy is reduced from £200 to £100 and the premium from £10 to £5, £100 sum assured and £5 premium will appear on the front of the sheet, and £200 sum assured and £10 premium on the back. It would, of course, be possible to deal only with differences, but this method would probably cause more mistakes.

The miscellaneous additions are carried direct to the Summary Book at the end of each year, but the miscellaneous deductions are combined with the corresponding cancelments and the totals carried to the Summary Book. The combination of cancelments and miscellaneous deductions is easily effected, as it is only necessary, when summarizing the several sets of deduction sheets for balancing purposes, to enter the particulars for each group in parallel columns, and then to sum them across, in order to obtain the combined figures required.

Taking the *Whole Life with Profits* class, for example, the deduction sheets are summarized and the results combined in the following manner:

Group	No. of Policies					Sums Assured					&c.
	D.	S.	L.	M.D.	Total	D.	S.	L.	M.D.	Total	
1815											
1816											
1817											
1818											
1819											
1820											
&c.											

the letters D, S, L, and M.D. indicating Deaths, Surrenders, Lapses, and Miscellaneous Deductions respectively.

This method of combining the cancelments and miscellaneous

deductions involves no additional labour, and simplifies the form of the final summary. When the combined deduction totals for each classification group have been carried to the Summary Book, they must be subtracted from the sum of the previous "totals in force", new policies, and miscellaneous additions, in order to obtain the new valuation particulars as at 31 December.

Automatic Alterations.

In addition to alterations which arise from changes in contract, correction of errors in age, and similar causes, there are also changes in some policies which occur automatically by flux of time. As explained in a previous section, the tab system provides a complete up-to-date record of automatic alterations, and the required particulars can be abstracted when the cards are scrutinized, the whole work being thus completed each year in one process. Incidentally, if the office does not grant endowment assurances for a shorter term than 10 years, it is unnecessary to turn over the cards relating to policies issued in the last 10 years when compiling a list of maturities for the current year. Such policies form, in most offices, a considerable proportion of the total number.

It is possible, of course, to keep the valuation cards in classification order instead of in numerical order, and to prepare the lists of maturities, &c., directly from the cards; but this does not necessarily solve the problem. For instance, if limited payment cases are valued by the year of birth method, the classification does not give the last payments falling due. Moreover, if the cards are kept in classification order, they do not so conveniently serve the purpose of a bonus record, and errors in sorting are more likely to occur than if numerical order is adhered to throughout.

The record of automatic alterations is sometimes kept in a special book containing the policy numbers of the cases relating to each calendar year. It must, of course, be written up from time to time, and soon becomes overcrowded with policies which are no longer in force. As the policy numbers only are recorded in the book, the required particulars must be obtained independently.

Final Valuation Particulars.

As previously mentioned, the initial "totals in force" are recorded in the Summary Book, and the new policies, additions,

and total deductions entered at the close of each year, the "totals in force" at 31 December being obtained by making the necessary additions and subtractions. In order to check this work at the final stage, it is advisable to summarize the policy particulars in force at the end of each year, as recorded by the Summary Book, and to reconcile the grand totals with the figures arrived at independently by adding the total new policies and miscellaneous additions for the year to the policy particulars in force at the beginning of the year and then subtracting the deductions *in toto*. The summary of policy particulars can also be used for the purpose of the actual valuation, the necessary factors having been inserted beforehand. It is, of course, desirable to value in advance the special class policies requiring individual treatment, in order to expedite the valuation results as much as possible.

In order to make the Summary Book last for several years without the necessity of re-writing, a whole page is devoted to each valuation group, and there is a separate Summary Book for each of the main classes of policies.

Additional Check on Results.

One advantage of the class-book system is that the policies comprised in each valuation group are set out in detail. Since, however, the tab system enables the valuation cards to be sorted into classification order very quickly, should it be desired to verify the Summary Book independently, the methods described are held to be free from objection on the ground that the individual policy particulars are so far lost in the Summary Book totals as to render the final results liable to suspicion—a criticism which might possibly be levelled against a system which failed to provide any facility for a speedy classification. In this connexion it may be suggested that although the precautions mentioned should keep the continuous totals in the Summary Book free from error, it will probably be desirable, as an additional safeguard, to extract the valuation cards of a particular class from time to time, and to verify the Summary Book figures by summing certain particulars on the cards. This process does not form an essential part of the valuation system, and need only be performed when convenient opportunities occur.

LEGAL NOTES.

By William Charles Sharman, F.I.A., *Barrister-at-Law.*

Validity of assignment of life assurance policy.

THE decision of Mr. Justice Astbury in the case of *In re Williams, Williams* v. *Ball*, reported in these Notes, *J.I.A.*, vol. l, p. 118, has now been affirmed by the Court of Appeal, 86 L.J. 36 C.A.

The case was concerned with the validity of the assignment of a policy of assurance, such assignment being in the form of an endorsement by the assured on the policy as follows: "I "authorize A.M.B. and no other person to draw this assurance "in the event of my predeceasing her." The facts are reported in the previous Note on the case.

In the course of the judgment Lord Cozens Hardy, M.R. said: "What, then, is the effect of the Memorandum here? It seems "to me to be of the nature of a mere power of attorney though "not under seal, authorizing the person named to do certain "things. A power of attorney, however, becomes inoperative "on the death of the person conferring it, and the recipient "cannot claim to exercise the power after that person's death."

"There is another objection to the document. It is a mere "mandate which ceased to be operative at death, and further "it seems to me to be, if anything, of the nature of a testa-"mentary document. It was a document which was intended "to take effect only in the event of the donor predeceasing "the donee.

"I think, therefore, that the decision of the learned Judge "in the Court below was right, and there was no gift of the "policy to the appellant."

Transfer of statutory deposit.

Fund to be ear-marked in respect to liabilities under paid-up policies.

The case of *In re City of Glasgow Life Assurance Co.*, 86 L.J. Ch. 86, deals with an interesting point arising on the amalgamation of two life assurance companies, and the proposed transfer of the fund deposited in Court under section 2 of the Assurance Companies Act from the vendor company to the purchaser company free from all liabilities of the vendor company. The facts are as follows:

The City of Glasgow Life Assurance Co. (afterwards called "the Glasgow company") was incorporated by a special Act in 1861, and further powers were conferred upon it by an Act

in 1892. In 1913 the company was registered as an unlimited company under the Companies (Consolidation) Act, 1908.

The Scottish Union and National Insurance Co. (afterwards called "the Scottish Union") was incorporated by a special Act in 1878, and further powers were conferred upon it by subsequent Acts; and by section 2 (H) of its special Act of 1892 it was empowered to carry into effect contracts for amalgamation with any company authorized to carry on any business which the Scottish Union was authorized to carry on or for undertaking all or any of the contracts and liabilities of any such company.

In 1910 each company deposited in pursuance of the requirements of section 2 of the Assurance Companies Act, 1909, securities representing £20,000.

By a provisional agreement made in March 1913, the Glasgow company agreed to sell, and the Scottish Union agreed to purchase as a going concern the entire undertaking, business, and assets of the Glasgow company, and to assume all the legal liabilities of that company; and the purchasing company agreed to keep separate and distinct the life assurance and annuity fund of the Glasgow company, and to continue to hold that fund in trust for the protection and sole benefit of the policyholders and annuitants of the Glasgow company.

By an Order made 16 October 1913 by the Lords of Council and Session in Scotland the transfer of the business of the Glasgow company to the Scottish Union in the terms of the agreement was duly sanctioned.

In November 1913, the Glasgow company went into voluntary liquidation, and in the following month the Glasgow company and its liquidators duly assigned its undertakings, business, and assets to the Scottish Union.

In December 1914, a petition was presented by both companies to the High Court of Justice in England, asking that the fund in Court to the credit "*Ex parte* the City of Glasgow Life Assurance Company in respect of life assurance business" might be transferred to an account "*Ex parte* the Scottish Union and National Insurance Company in respect of life assurance business," and that the income of the funds might be paid to the Scottish Union. No order was made on that petition owing to objections raised by the Board of Trade, except with regard to the payment of dividends on the fund to the Scottish Union.

In respect of a large number of policies issued by the Glasgow

company premiums thereon had, subsequent to the amalgamation of the two companies, been paid by the policyholders to the Scottish Union, in consequence whereof there had been a novation of the contracts with the policyholders. But there were, on the other hand, a large number of fully paid-up life policies remaining on foot, with regard to which novation of the contracts was not probable for a long time to come.

It was proposed, with the consent of the Board of Trade, that claims of these paid-up policyholders should be protected by the carrying over the fund to the account of the Scottish Union so ear-marked that it shall remain available for those policyholders who have not accepted policies with the Scottish Union.

In delivering judgment, Sargant, J., said:

"It appeared that, as regards a very large number of policies, "annual or other periodical payments had been made to the "purchaser company; and with regard to those policies it was "probable or certain that there had been, by reason of that "payment, a novation of the contract with the policyholder, "so that the purchaser company became liable instead of, and to "the release of, the vendor company, for section 7 of the Life "Assurance Companies Act, 1872, had ceased to be law and a "novation was, therefore, possible in that way. But beyond "these policies, the vendor company had issued a very large "number of fully paid-up policies, the consequence of which was "that there had not been, and probably would not be for a very "long time to come, a novation of these contracts between the "original policyholders and the vendor company.

"Under these circumstances it was suggested that the decision "of Mr. Justice Warrington in Popular Life Assurance Co. "(*J.I.A.*, vol. xliii, p. 224) applied; and that, the vendor company "having actually been dissolved and there being no personal "remedy at all against the vendor company, it necessarily "followed that the remedy against the sum deposited in Court "had also gone, and that, therefore, the purchaser company was "entitled to have that fund transferred to it as the assignee of "the vendor company without any provision at all being made "for those fully paid-up policies of the vendor company. I "do not think that that view of the case is really right, or that "Mr. Justice Warrington meant to go to that extent. I have "had the papers sent to me by the Board of Trade, and I find "that in the petition in that case it was stated that the purchaser "company there had undertaken the liabilities of the vendor

"company on all policies issued by such company, and that the "purchaser company had become liable on these policies in the "same way as the vendor company was liable thereunder. I "think it must have been on that basis that the Order was made "by Mr. Justice Warrington. Of course it is familiar that the "Court, at any time within two years of the dissolution of the "company in a voluntary winding-up, could put an end to the "dissolution; and even apart from that, it is clear again that "the assets of the company, which is voluntarily wound up, "cannot be transferred to a purchaser without providing, not "merely for present liabilities, but for future possible liabilities, "such as covenants under leases, and so on. Therefore, in such "a case as this, as long as there are existing liabilities of a vendor "company, it seems to me that it would not be right for the "Court to direct the transfer of the assurance fund to the pur-"chaser company free from these possible liabilities of the "vendor company.

"But since the petition first came before me, a suggestion "has been made by the Board of Trade which seems to me amply "to protect those policyholders who have not novated—namely, "that the fund shall be carried to a special account which will "on the face of it show that the fund is still applicable for "meeting those liabilities in the possible, though perhaps very "improbable, event of the policyholders having to resort to it. "Accordingly I make the order which has been asked for to that "effect."

Statutory deposit available for costs of winding-up.

Another case dealing with the Statutory deposit required to be made by assurance companies is that of *In re National Standard Life Assurance Corporation (Limited)*, 33 T.L.R. 65.

This company was incorporated in 1906, and paid into Court a deposit of £20,000 now represented by certain securities in the hands of the Accountant-General of the Court of Chancery.

On 6 June 1916 the company was ordered to be wound up and a liquidator appointed.

It appeared that the balance available for the general purposes would be small and it was uncertain whether beyond the deposit in Court any substantial assets would be realized in the liquidation.

It was doubtful whether the claims made by persons claiming

under policies of assurance issued by the company would not exceed the amount of the deposit.

The liquidator now applied to the Court for directions whether the securities representing the £20,000 deposit formed part of the general assets of the company so as to be available for

(1) The general cost of the winding up, including the getting in of the general assets of the company.

(2) Moneys payable into Court as and when ordered by way of costs in pending actions by the company with which the liquidator had been authorized by the Court to proceed.

Mr. Justice Neville made the following order:

That the securities representing the £20,000 deposit are available for the general costs of the winding up so far as they relate to the life assurance business of the company and to getting in such of the assets as relate to that business and for the payment of money paid and payable into Court as security for costs in actions by the company relating to life assurance business and with which the liquidator is authorized by the Court to proceed.

ACTUARIAL NOTES.

On a method of determining approximately the rate yielded by a redeemable Bond. By J. R. ARMSTRONG, F.F.A., *of the Scottish Provident Institution.*

IN their Actuarial Note (*J.I.A.*, vol. xlix, p. 366), Mr. Lidstone and Mr. Todhunter make use of the assumption that "for a small variation, say x, in the rate (of interest) we have $\frac{1}{a^{i+x}}-\frac{1}{a^{i}}=\theta x$", where θ varies so slowly that for approximate calculations it may be treated as a constant.

This assumption enables us to approximate directly, and within limits very closely, to x, the difference between the actual yield $(g+x)$ and the nominal rate g on a redeemable bond bought at a discount of K per unit. For

$$x=\frac{\mathrm{K}}{a^{g+x}}=\mathrm{K}\left[\frac{1}{a^{g}}+\left(\frac{1}{a^{g+x}}-\frac{1}{a^{g}}\right)\right]=\mathrm{K}\left(\frac{1}{a^{g}}+\theta x\right)$$

on the assumption in question. Hence (denoting a^{-1} by R for convenience)

$$x=\frac{\mathrm{KR}^{g}}{1-\mathrm{K}\theta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the case of a bond bought at a *premium* of K per unit we have, substituting $g-x'$ for $g+x$

$$x' = \frac{\mathrm{KR}^n}{1+\mathrm{K}\theta} \quad . \quad . \quad . \quad . \quad (2)$$

If, however, instead of adopting the constant θ for all rates and terms we substitute for it $100\left(\frac{1}{a^{g+\cdot 01}} - \frac{1}{a^g}\right)$ we obtain a formula almost as simple, but somewhat more flexible and thus applicable with confidence over a wider area. Writing R as before for a^{-1} and making the suggested substitution, we have for a *discount* of K per unit,

$$100x = \frac{100\mathrm{KR}^n}{1-100\mathrm{K}(\mathrm{R}^{g+\cdot 01} - \mathrm{R}^g)} \quad . \quad . \quad . \quad (1a)$$

making the yield $100(g+x)$ per-cent; and for a *premium* of K per unit,

$$100x' = \frac{100\mathrm{KR}^n}{1+100\mathrm{K}(\mathrm{R}^g - \mathrm{R}^{g-\cdot 01})} \quad . \quad . \quad . \quad (2a)$$

making the yield $100(g-x')$ per-cent.

The main point in either pair of formulas is that a^{-1} in the numerator is always taken at the nominal rate, which may of course vary from, say, 3 to 6 per-cent or more. Most cases in practice would, however, be covered, and the use of the formulas greatly facilitated by a working table of three columns such, for example, as the following, of which the central column contains the values of $\frac{2}{a_{2n}}$ at 2 per-cent per half-year, and the left and right-hand columns the differences of this quantity for half-yearly rates one-half per-cent lower and higher respectively:

Years (n)	Difference for 1 less	R4	Difference for 1 more
⋮			
10	·00582	·12231	·00598
⋮			
30	·00675	·05754	·00717
⋮			
50	·00766	·04641	·00822
⋮			

Example 1.—Price 88·842, term 10 years, nominal rate $4\frac{1}{2}$ per-cent, payable half-yearly. Here $100\mathrm{K}=11{\cdot}158$, and taking $\mathrm{R}^{4\frac{1}{2}}$ in the numerator (to four places) at

$\cdot 1223+\cdot 0030=\cdot 1253$ and $\Delta\mathrm{R}$ in the denominator at $\cdot 0060$, we have, by ($1a$)

$$100x=\frac{11\cdot 16\times\cdot 1253}{1-11\cdot 16\times\cdot 0060}=1\cdot 498 \quad \text{whence} \quad 100j=100(g+x)=5\cdot 998$$

the true rate being 6 per-cent.

Example 2.—Price 129·731, term 50 years, nominal rate 6 per-cent, payable half-yearly. Here $100\mathrm{K}=29\cdot 731$, and taking R^{6} by inspection $=\cdot 0464+(\cdot 0082+\cdot 0087)=\cdot 0633$ say, and $\Delta\mathrm{R}^{g-\cdot 01}=\cdot 0087$, we have, by ($2a$)

$$100x'=\frac{29\cdot 73\times\cdot 0633}{1+29\cdot 73\times\cdot 0087}=1\cdot 495 \quad \text{whence} \quad 100j=100(g-x')=4\cdot 505$$

the true rate being $4\frac{1}{2}$ per-cent.

Where income tax (at, say, t per £ on g) is taken into consideration, the formula is specially suitable. For a bond bought at a discount of K per unit, we have

$$100x(1-t)=\frac{100\mathrm{KR}^{g-tg}}{1-100\mathrm{K}(\mathrm{R}^{g-tg+\cdot 01}-\mathrm{R}^{g-tg})}$$

or, if at a premium,

$$100x'(1-t)=\frac{100\mathrm{KR}^{g-tg}}{1+100\mathrm{K}(\mathrm{R}^{g-tg}-\mathrm{R}^{g-tg-\cdot 01})}$$

the result in either case being the difference, positive or negative, between the *net* yield $j(1-t)$ and the *net* nominal rate.

For an example, take the first one given in the Note above referred to, where $n=10$, $\mathrm{K}=\cdot 15684$, $g=\cdot 06$, and $t=\cdot 05$. Here we have

$$100x'(1-\cdot 05)=\frac{15\cdot 684\mathrm{R}^{5\cdot 7}}{1+15\cdot 684(\mathrm{R}^{5\cdot 7}-\mathrm{R}^{4\cdot 7})}$$

Taking R and $\Delta\mathrm{R}$ by inspection from the above table at $\cdot 1325$ and $\cdot 0060$ respectively, we have $100x'(1-\cdot 05)=1\cdot 899$ and the *net* yield $5\cdot 7-1\cdot 899=3\cdot 801$, or $4\cdot 001$ subject to tax, the true yield subject to tax being 4 per-cent.

As a rule, sufficiently approximate results will be obtained by the formulas even when x lies well outside the 1 per-cent interval for which $\Delta\mathrm{R}$ is taken. In the more extreme cases, where n, K and x are large, the mean of the differences for two, and (more rarely) even three, such intervals would of course secure a closer approximation.

Where yields are frequently required, Deghnée's or other tables are no doubt available, and the present formulas and suggestions are not put forward as an alternative. In other cases these formulas may be found serviceable, while, apart

from their practical utility, the student of the theory of interest may possibly find them in some respects helpful.

[The formula discussed by Mr. Armstrong is, in effect, a first-difference interpolation formula, and will be found to be a particular case of formula (14), p. 116, *Text-Book,* Part I (Revised Edition)*—g and $g+\cdot01$ being taken as trial rates. It has the advantage of simplicity, and since the first difference of the reciprocal of the annuity-value changes very slowly with the rate of interest, the fact that in many cases g would not appear to be a suitable trial rate is not of much importance. It has, however, been pointed out to us by Mr. Lidstone that, consistently with the author's general method, a somewhat closer approximation would be obtained by taking as the trial rate a standard rate I—say, at the present time, 5 per-cent—instead of g. The formula would then become

$$\frac{i-\mathrm{I}}{h}=\frac{g-\mathrm{I}+\mathrm{KR}^{\mathrm{I}}}{h-\mathrm{K}\Delta\mathrm{R}^{\mathrm{I}}}$$

where h is the interval of differencing; and (for a given value of n) R and ΔR would vary only at comparatively long intervals with the current yield instead of from bond to bond with g.

With regard to these (and other) first difference interpolation methods it may be suggested that in practice it is probably simpler and safer to dispense with formulas and to interpolate for A, i or g, as the case may be, directly from the well-known expressions for these quantities. The most convenient plan—especially when Bond Tables are available—is to interpolate for A (the value of the bond), but greater accuracy, if required, is obtainable by interpolating for g, as in the formula under discussion.

It may be added, as a matter of historical interest, that a similar formula—with h instead of ·01 as the interval of differencing—was given by the late Sir George Hardy in a paper on "Compound Interest Tables and Formulas", read before the Actuarial Society of Edinburgh in December 1890. This paper was not published, and its contents—with the exception of the formula for the rate of interest earned on an assurance fund—do not seem to have been generally known. We understand that the MS has been discovered among Sir George Hardy's papers, since we received Mr. Armstrong's note.—EDS. *J.I.A.*].

* Formula (22), p. 119 in the previous edition.

Graduation by Makeham's Hypothesis. By W. PALIN ELDERTON, F.I.A., *and* S. J. ROWLAND, F.I.A.

VARIOUS attempts have been made from time to time to make graduations on the assumption that the force of mortality or that colog p_x takes the form $A+Bc^x$ and these attempts have often been unsuccessful, because the calculations have not taken properly into account the number of observations from which the various values of the force of mortality were obtained. The choice of specified equidistant values, for instance, left out of account a large part of the experience, and when several trials with various different sets are made the results are troublesome to combine and are uncertain as regards accuracy. The best of the earlier methods was that due to King and Hardy, and consisted of finding the constants A, B and c from four summations of $f(x)$, namely, (1) from $f(x)$ to $f(x+a-1)$; (2) from $f(x+a)$ to $f(x+2a-1)$; (3) from $f(x+2a)$ to $f(x+3a-1)$; and (4) from $f(x+3a)$ to $f(x+4a-1)$. The extreme values would usually be left out altogether. The objections to the method are that the four sums are not of equal value, (1) and (4) being worth less than (2) and (3), and proper weight is not given to the various items in each summation. The principle of the method might be retained and the objections to a large degree be removed by using the following summations:

$$(1)\quad hf(a)+kf(a+4)+lf(a+8)+\ldots$$

$$(2)\quad hf(a+1)+kf(a+5)+lf(a+9)+\ldots$$

$$(3)\quad hf(a+2)+kf(a+6)+lf(a+10)+\ldots$$

$$(4)\quad hf(a+3)+kf(a+7)+lf(a+11)+\ldots$$

where h, k, l, &c., are coefficients roughly based on the exposed to risk. So far as we know this method has not been used except in Note G, p. 131, of Sir George Hardy's "Construction of Mortality Tables", and the recent tendency has been to discard the King and Hardy method and adopt either the method of moments applied to $A+Bc^x$ or some plan which works directly from the exposed to risk and deaths or by some other method gives appropriate value to the various parts of the table. The application of the method of moments directly to $A+Bc^x$ is open to similar objections to those urged against the King and Hardy method, and when, as is sometimes done, it is employed in a form that necessitates the assumption of the value of c it is open to further objection. This method

may give good results, but it deserts the facts, and is not the best method available.

The methods that take some account of the facts from which the ungraduated values of colog p_x are found, are*:

(1) G. F. Hardy's method of working from the exposed to risk and deaths, assuming, after trial, a value for c; this method was used for the British Offices Tables†;

(2) Application of the method of least squares used by Karup and others, and by Steffensen in the recent Danish experience: the various writers employ different methods:

(3) Steffensen's method using least squares, but weighting with c^{-x} as an approximation; this is given in a recent publication;

(4) The rough representation of the exposed to risk and the graduation of a new series of deaths calculated from the hypothetical exposed to risk.

The various methods have been fully described in the books and papers mentioned in the list appended to this note, and we need not deal with detail of the application here, but we may give a table based on that shown by Dr. Steffensen in one of his recent papers exhibiting the results of various attempts to graduate the Danish experience, and we have added a graduation by method (1) assuming that $\log c = \cdot 04$. We have reproduced the results of a graduation made by Steffensen, resulting from the application of the method of moments to $A + Bc^x$, which compares unfavourably with the other methods as, in fact, would be expected.

As regards method (4) we give the results of a graduation reached by the assumption of a normal frequency curve as the hypothetical exposed to risk and the graduation of the figures resulting from multiplying this exposed by colog p_x. The method has been fully described, but it will perhaps be of some interest to show the whole of the arithmetical work, and we therefore give it in Appendix II.

We may now give the tables showing the results of the various graduations‡:

* H. P. Calderon indicated some methods in *J.I.A.*, vol. xxxv, pp. 15 *et seq.*, but the general statement given here covers the methods suggested.

† The method was suggested by Makeham, *J.I.A.*, vol. xvi, pp. 344 *et seq.*, for use with imperfect data, to the extent of an assumed log c with actual deaths and exposed, but not successive summations.

‡ We have, of course, assumed the arithmetical accuracy of all the results given by Dr. Steffensen.

TABLE I.—*Values of constants for colog p_x.*

No.	Method	A	log B	log c
(1)	G. F. Hardy's method with exposed and deaths...	·0009823	$\bar{5}$·63911	·04
(2)	Steffensen—Least squares, correctly weighted...	·0009033	$\bar{5}$·66499	·039668
(3)	Steffensen—Least squares, weighted, c^{-x}	·0015200	$\bar{5}$·81270	·037377
(4)	Hypothetical exposed—normal curve of error ...	·0006184	$\bar{5}$·77119	·038233
(5)	Moments applied to A + Bc^x	·0006242	$\bar{1}$·00369	·035203

TABLE II.—*Specimens of Rates of Mortality.*

Age	(1)	(2)	(3)	(4)	(5)
20	·00289	·00274	·00433	·00221	·00261
25	·00327	·00312	·00478	·00265	·00319
30	·00384	·00372	·00546	·00333	·00407
35	·00479	·00467	·00652	·00437	·00539
40	·00625	·00617	·00814	·00601	·00735
45	·00856	·00853	·01064	·00853	·01030
50	·01223	·01225	·01446	·01247	·01470
55	·01800	·01809	·02030	·01848	·02125
60	·02708	·02725	·02922	·02779	·03101
65	·04131	·04152	·04278	·04206	·04546
70	·06343	·06364	·06326	·06382	·06673
75	·09748	·09752	·09390	·09666	·09774
80	·14886	·14854	·13908	·14533	·14233
85	·22447	·22328	·20419	·21580	·20510
90	·33058	·32816	·29485	·31391	·29075
95	·47015	·46568	·41452	·44249	·40221

TABLE III.—*Expected Deaths.*

Age Group	Actual	Expected by (1)	Expected by (2)	Expected by (3)	Expected by (4)	Expected by (5)	Expected (2) less Actual +	Expected (2) less Actual −	Expected (4) less Actual +	Expected (4) less Actual −
20–	5	2	2	3	2	2	...	3	...	3
25–	22	15	14	22	12	15	...	8	...	10
30–	96	88	85	122	78	96	...	11	...	18
35–	214	226	222	300	212	259	8	...	...	2
40–	310	359	356	455	352	427	46	...	42	...
45–	459	467	466	569	457	565	7	...	...	2
50–	563	549	549	636	560	653	...	14	...	3
55–	648	613	618	674	621	717	...	30	...	27
60–	663	659	663	698	671	745	...	0	8	...
65–	693	678	681	691	686	725	...	12	...	7
70–	628	614	615	601	613	626	...	13	...	15
75–	432	448	448	425	443	447	16	...	11	...
80–	240	252	251	232	244	237	11	...	4	...
85–	80	82	81	74	77	74	1	...	...	3
90–	12	13	13	12	12	11	1	...	...	0
Total	5,065	5,065	5,064	5,514	5,010	5,599	90	91	65	90

Method (3) is unsuccessful, probably because the approximation to the weights is inaccurate at the early ages; Steffensen, however, found that when the ends of the table were neglected his results were good. Methods (1) and (2) give almost identical results, but (2) is slightly better. Method (4) is worse in the first three groups and better afterwards.

The various graduations given above and the other graduations by Steffensen of parts of the table have made us doubt whether Makeham's hypothesis is, after all, entirely satisfactory for the statistics throughout. If we choose constants to fit the start, the curve diverges from the facts elsewhere, and if we graduate the data satisfactorily from 35 the earlier part is open to criticism. Slight evidence in support of this view can be found in Steffensen's least squares graduation (2) where we notice that roughly speaking the mortality is too low from 20 to 32, too high from 32 to 47, too low from 47 to 72, and too high from 72. Stretches of such divergencies sometimes indicate unsuitable curves. It must not be inferred that we think Steffensen's graduation is unsatisfactory if a Makeham hypothesis is to be assumed throughout; it is not; all we wish to urge is that a Makeham curve above 35 modified below that age would be a nearer representation of the mortality indicated by the statistics. We should ourselves be tempted to write ·0046 for the rate of mortality for all the ages up to age 35, and then use the rates of mortality given by method (4). The expected deaths for the first four groups would then be 3, 18, 95, 213, and the plus and minus deviations are both 65. The objection to such modifications that well-known approximations cannot then be used conveniently is hardly tenable; the error in assuming equivalent ages on Makeham's hypothesis below age 35, would be less important than the error in the graduation. If Makeham's hypothesis is not applicable in any particular case, it would seem preferable to have the rates of mortality undistorted and to bring in the error caused by the assumption of that hypothesis only in the comparatively few cases in which the approximation is helpful rather than have the rates of mortality in error, so that inaccuracy is present even in the many cases in which the hypothesis is of no assistance.

We may conclude these notes by saying that in practical graduation it seems to us important to bear in mind, first, that any graduation that operates only on a rate of mortality, or other similar function, may in some circumstances lead us far

from the facts, and it is well to work in such a way as to take into account the amount of data on which each individual rate is based, and secondly, that we may not be able to obtain a satisfactory graduation of certain tables by a particular formula, however excellent the method of application, because the formula itself is inapplicable.

*** May I take this opportunity of saying that I agree with Dr. Steffensen's general remarks, about the method of moments, in *J.I.A.*, vol. xlix, pp. 355 *et seq.*? I should not expect to obtain very satisfactory results from it unless I was operating on the actual number of cases: rates or measures derived therefrom may, sometimes, be graduated satisfactorily by moments or by unweighted least squares, but I should not expect satisfaction generally. I must confess to an inability to acquire any liking for the method of least squares, either in theory or practice, and it is probably this unreasonable dislike that leads me to ask if it can be proved generally that "least squares" give the "best" result. I do not think such a proof exists, or perhaps, can ever exist. It would of course involve a definition of "best" and as a further indication of the difficulties involved I may remark that a mean found from the original ungraduated facts may not be the "best" mean that those facts should imply.—W. P. E.

APPENDIX I.

References to publications giving information as to formulæ, &c., by which graduations (1) to (5) are obtained.

J. F. Steffensen.

"On the fitting of Makeham's curve to Mortality Observations." Math. Congress. 1912. (2).

"On the graduation of Mortality Tables by G. F. Hardy's modification of the Method of Moments." Svenska Aktuarieföreningens Tidskrift. 1915. (5).

"On graduation by Makeham's formula." Do. 1916. (3).

G. F. Hardy.

"British Offices Life Tables—Graduation of Without-Profit Mortality Table." *J.I.A.*, vol. xxxviii, pp. 501 *et seq.* (1).

"Construction of Mortality Tables." Layton. 1909. Pp. 63 *et seq.* (1) and (4). Note G, pp. 131 *et seq.*

W. P. Elderton.

"Frequency Curves and Correlation." Pp. 98 *et seq.* (4).

G. J. Lidstone.

"Further remarks on the Valuation of Endowment Assurances in Groups." *J.I.A.*, vol. xxxviii, p. 11. (5).

APPENDIX II.

Numerical Example of Graduation by Makeham's Hypothesis assuming a Normal Curve of Error for the Exposed to Risk.

The method is described on pp. 99 and 100 of "Frequency Curves and Correlation." The first step is to choose a suitable normal curve of error for the exposed to risk. It is not necessarily the normal curve that would result from the fitting of a normal curve to the actual exposed, because such a curve would generally give too large an exposed beyond the old-age limit of the table, and consequently lead to an incomplete curve in the resulting figures which are to be found by multiplying the hypothetical exposed to risk by colog p_x. The alternative would be to assume artificial values of colog p_x after the table ends—an unsatisfactory procedure.* We chose, therefore, a normal curve with a standard deviation of 10 and an origin at 52½. This meant that we had to take out of Sheppard's tables every tenth value of Z (the ordinate) and multiply by the appropriate ungraduated value of colog p_x. The ungraduated values are given in one of Steffensen's papers, and, to show the procedure exactly, we may give the following values:

Age	30	40	50	51	52	53	60	70	80
colog p ...	·00194	·00185	·00549	·00607	·00671	·00735	·01150	·02721	·06207

The table of $Z \times \text{colog}\ p_x$ was formed by Crelle, and all the work is given below. Each Z applies to two ages, and we give the values of Z, although in practice there is no need to copy them out of Sheppard's tables:

* G. F. Hardy gave formulæ for assuming a hypothetical Type III curve. If we had large tables so that suitable curves could be picked out, as they can from tables of the probability integral, this would improve the method, but a Type III curve fitted to the exposed leads to the same difficulty as a fitted normal type. It also means that the ordinates of the Type III curve have to be worked out, and the process therefore becomes very laborious.

$10^5 \times Z \times \text{colog}\, p_x$	x	Z	x	$10^5 \times Z \times \text{colog}\, p_x$	$10^5 \times Z \times \text{colog}\, p_x$	x	Z	x	$10^5 \times Z \times \text{colog}\, p_x$
267	52	·3984	53	293	6·5	31	·03955	74	167
239	51	·3945	54	258	6·1	30	·03174	75	145
212	50	·3867	55	303	4·5	29	·02522	76	136
191	49	·3752	56	394	4·2	28	·01984	77	102
143	48	·3605	57	300	1·9	27	·01545	78	79
154	47	·3429	58	422	9·2	26	·01191	79	58
128	46	·3230	59	333	...	25	·00909	80	56
109	45	·3011	60	346	3·5	24	·00687	81	42
81	44	·2780	61	392	1·3	23	·00514	82	47
75	43	·2544	62	351	...	22	·00384	83	38
74	42	·2299	63	400	1·2	21	·00279	84	26
49	41	·2059	64	303	...	20	·00203	85	15
34	40	·1826	65	335	...	...	·00146	86	27
28	39	·1604	66	309	...	...	·00104	87	9
38	38	·1394	67	370	...	...	·00073	88	12
34	37	·1200	68	294	...	...	·00051	89	9
20	36	·1023	69	241	...	...	·00035	90	6
17	35	·0863	70	235	...	...	·00024	91	1
18	34	·0721	71	224	...	...	·00016	92	1
11	33	·0596	72	217	...	...	·00011	93	4
9·4	32	·0488	73	196	...	...	·00007	94	...
					...	...	·00005	95	
					...	...	·00003	96	1 say
					...	...	·00002	97	
					...	...	·00001	98	..

To save the labour of multiplying each term we then summed them in fives and formed the following table and calculated the moments about the centre of the 50– group, *i.e.*, about 52. The figures are as follow :—

	$y = 10^5 \times Z \times \text{colog}\, p$	Distance (d) of Group from 52	$y \times d$	$y \times d^2$
20–	6	−6	36	216
25–	20	−5	100	500
30–	51	−4	204	816
35–	134	−3	402	1,206
40–	313	−2	626	1,252
45–	725	−1	725	725
50–	1,269	0	−2,093	
55–	1,752	+1	1,752	1,752
60–	1,792	+2	3,584	7,168
65–	1,549	+3	4,647	13,941
70–	1,036	+4	4,144	16,576
75–	520	+5	2,600	13,000
80–	209	+6	1,254	7,524
85–	72	+7	504	3,528
90–	15	+8	120	960
95–	1	+9	9	81
	9,464		+18,614	69,215
			+16,521	or 7·316673 per unit frequency
			or 1·745668 per unit frequency	

This grouping necessitates an adjustment in the second moment which has to be reduced by ·08 to allow for the assumption that the sum of the five ordinates can be treated as if they are concentrated at the mid ordinate. The adjustment is less than that (·083) given by Sheppard, when *areas* are assumed to be concentrated at the mid ordinate.* The moments (adjusted) about 52, are therefore

1st moment ..	1·745668
2nd	7·236673

These have to be transferred to $52\frac{1}{2}$, the origin of the normal curve assumed for the exposed, and made so that the unit of grouping is 1 year instead of 5 years. The latter adjustment means multiplying the 1st moment by 5 and the second by 25. The transference means the deduction of $\frac{1}{2}$ from the multiplied 1st moment and $\left(\text{the multiplied 1st moment} - \frac{1}{4}\right)$ from the multiplied 2nd moment: hence

$$\mu'_1 = 1\cdot745668 \times 5 - \cdot5 = 8\cdot22834$$

$$\mu'_2 = 7\cdot236673 \times 25 - 8\cdot72834 + \cdot25 = 172\cdot43850$$

The rest of the work merely means the substitution of numerical values in the expressions on pp. 99 and 100 of "Frequency Curves and Correlation."

$$t - h = \frac{\mu'_2 - \sigma^2}{\mu'_1} = \frac{72\cdot43850}{8\cdot22834} = 8\cdot803537$$

$$\log_{10} c = \frac{t-h}{\sigma^2} \log_{10} e = \cdot08803537 \times \cdot4342945 = \cdot03823327$$

$$\mathrm{N}_2 = \frac{\mu'_1(\mathrm{N}_1 + \mathrm{N}_2)}{t-h} = \frac{8\cdot22834 \times 9464}{8\cdot803537} = 8845\cdot65$$

$$\mathrm{N}_1 = 9464 - 8845\cdot65 = 618\cdot350$$

$$\mathrm{A} = \mathrm{N}_1 \div 10^6 = \cdot00061835$$

$$\log_{10}\mathrm{B} = \log \frac{\mathrm{N}_2}{10^6 c^{\frac{t+h}{2}}} = \log 8845\cdot65 - 6 - 56\cdot90177 \times \cdot03823327$$

$$= \bar{5}\cdot771189$$

* It is interesting to remark that unadjusted moments would have reproduced Steffensen's least squares graduation almost exactly—log c being ·0392889, A ·0008560, and log B $\bar{5}$·69452.

The 10^6 arises because we took 10^5 times Z when multiplying by colog p_x and in integrating the normal curve a σ, which is also 10, has to be brought into account.

It is interesting to note that if we use this method and assume the values of log c given by methods (1) and (2) in the table on p. 253, we obtain the following values of A and log B:

	A	log B	log c
(1)	·001009	5·64291	·04
(2)	·0009383	5·66700	·039668

It will be seen that these values approximate to those given in the earlier table, and only one moment is used. This also points to the fact that the value given to log c is very important.

Summation Formulæ with Second Difference Errors. *By* A. D. Watson, A.I.A.

It was shown by Mr. George King (*J.I.A.*, vol. xli, p. 71) that if a summation formula which introduces an error of $+\frac{1}{12}\frac{d^2}{dx^2}$ is applied to colog p_x of a Makeham curve the errors introduced on the one hand by the second differential coefficient and on the other by the neglect of the fourth and higher differential coefficients tend to balance, owing to the fact that $\frac{d^2}{dx^2}, \frac{d^4}{dx^4}, \frac{d^6}{dx^6} \ldots\ldots$ are all positive, while the coefficients of $\frac{d^4}{dx^4}, \frac{d^6}{dx^6} \ldots\ldots$ are negative; and Mr. Spencer subsequently devised a formula (*J.I.A.*, vol. xli, p. 404), in which these two errors very nearly neutralize each other.

It may be noted that the small theoretical error introduced by a formula of the class in question, *e.g.*, by Hardy's well-known Friendly Society formula, can be eliminated by combining in the proper proportions colog p_x graduated by that formula with colog p_x graduated by a formula correct to third differences, the introduced errors being in opposite directions.

The resulting curve would, of course, be a smooth curve, and the smoothing coefficient of the combined graduation formula would be a proportionate combination of the smoothing coefficients of the individual formulæ. It would be unsatisfactory to combine a graduation by the Friendly Society formula with one by Woolhouse's formula as thereby the greater smoothing power of the former would be materially reduced.

An examination of Mr. Spencer's Comparative Table (*J.I.A.*, vol. xli. p. 390) suggests that either of Higham's formulæ

$$(a)\quad \frac{1}{1875}\left\{12[5]^4-25[5]^2[9]\right\}u_0$$

or $$(b)\quad \frac{[5]^3}{125}\left\{u_0-b_{-1}-b_0-b_{+1}\right\}$$

might be suitable for combining with Hardy's. Both of Higham's formulæ have the same range as Hardy's, while Higham's (a) has nearly the same smoothing power as Hardy's, but is somewhat cumbersome to apply. Higham's (b), while inferior to Hardy's in smoothing power, is very suitable for combining therewith as the operand is the same, and one summation is common to each, so that the two graduations may be performed with an inconsiderable amount of additional work. If a graduation not involving more than 17 terms, and yet free from introduced error, is desired, the foregoing method might in some cases be found satisfactory. The advantages in practice, if any, would not probably be important.

Any formula introducing a negative error might, of course, be similarly combined with Hardy's or any other formula introducing a positive error. In the Table A, following, are shown the errors introduced by a few well-known formulæ, including two new formulæ, Nos. 7 and 8, being respectively Dr. Karup's and Mr. Spencer's original 21-term formula modified so as to introduce a second difference error. From the error factors shown in the table may readily be deduced the proportions in which colog p_x derived by Hardy's formula should be combined with colog p_x derived by any other formula so as to eliminate the introduced error. The proportions differ according to the value of c, but for practical purposes the greater portion of the error would be eliminated in most tables by assuming log c = ·039 or ·04.

TABLE A.

Classification of formulæ showing the value of e where $\epsilon\beta e^x$ is the error introduced by applying the formula to graduate $(a+\beta e^x)$ = colog p_x.

No.	Author	Formula	Error Factor, e: $\log_{10}e = \cdot039$	Error Factor, e: $\log_{10}e = \cdot04$
1	J. A. Higham 17-term ...	$\frac{1}{1875}\{12[5]^4-25[5]^2[9]\}u_0$	−·0003963	−·0004384
2	" " ...	$\frac{[5]^3}{125}\{u_0+u_{\pm1}-u_{\pm2}\}$	−·0001229	−·0004678
3	G. F. Hardy 17-term ...	$\frac{[4][5][6]}{120}\{u_0+u_{\pm1}-u_{\pm2}\}$	+·0002423	+·0002317
4	Karup 19-term	$\frac{[5]^4}{625}\{3u_0+3u_{\pm1}-2u_{\pm3}\}$	−·0005165	−·0005713
5	J. Spencer original 21-term	$\frac{[7][5]^2}{350}\{2u_0+u_{\pm1}-u_{\pm3}\}$	−·0008726	−·0009655
6	J. Spencer new 21-term...	$\frac{[4][5]^2[6]}{600}\{3u_0-u_{\pm3}\}$	−·0000122	−·0000501
7	Karup 19-term modified	$\frac{[4][5][6]}{600}\{3u_0+3u_{\pm1}-2u_{\pm3}\}$	+·0001186	+·0001279
8	J. Spencer original 21-term modified	$\frac{[4][6][7]}{336}\{2u_0+u_{\pm1}-u_{\pm3}\}$	−·0001728	−·0002279

In practice a 21-term formula, or even a 19-term formula, would not be combined with Hardy's formula in the foregoing manner; for it would be preferable to use Mr. Spencer's new 21-term formula, No. 6 in the above Table, which is almost free from error in graduating colog p_x for any usual value of c, and is of higher smoothing power than any such combined formula would be; or, if a 19-term formula were desired, No. 7, which introduces about half as much error as Hardy's and is of high smoothing power, would probably be found quite as satisfactory as any combined formula.

Smoothing power is the first consideration in determining the value of a formula. Mr. King states that in the case of Hardy's formula the error introduced in the graduation of colog p_x is not of practical importance. It occurs mainly at the advanced ages. It will be noted from the foregoing Table that formula

No. 8, which also involves a second difference error, generally introduces less error into colog p_x than Hardy's—approximately 75 per-cent when $\log c = \cdot 039$—the error being, however, in the opposite direction. In comparison with Mr. Spencer's new 21-term formula the error varies from about 15 times to about $1\frac{1}{2}$ times, as $\log c$ varies from ·039 to ·01, but as the error introduced by Mr. Spencer's new formula is practically nil the error introduced by formula No. 8 is not of great importance. The smoothing power of formulæ Nos. 6, 7 and 8 is compared in the following table in the manner adopted by Mr. Spencer (*J.I.A.*, vol. xli, p. 390).

No.	Formula	Fourth Difference Error coefficient of $-\frac{d^4}{dx^4}u_0$	Sixth Difference Error coefficient of $-\frac{d^6}{dx^6}u_0$	Sum of coefficients of 5 central terms	Sum of Third Differences irrespective of sign of coefficients entering into formula	Smoothing coefficient
6	$\frac{[4][5][6]}{600}\{3u_0 - u_{\pm 3}\}$	10·30	27·25	·793	·166	$\frac{1}{141}$
7	$\frac{[4][5][6]}{600}\{3u_0 + 3u_{\pm 1} - 2u_{\pm 3}\}$	7·90	18·05	·826	·116	$\frac{1}{120}$
8	$\frac{[4][6][7]}{336}\{2u_0 + u_{\pm 1} - u_{\pm 2}\}$	12·70	35·81	·756	·107	$\frac{1}{172}$

The coefficients of these formulæ written in extenso are as follows:

No. 6. $\frac{1}{600}(-1, -4, -7, -8, -5, 6, 27, 54, 81, 102, 110, 102 \ldots)$

No. 7. $\frac{1}{600}(-2, -6, -9, -8, 2, 23, 52, 83, 107, 116, 107 \ldots)$

No. 8. $\frac{1}{336}(-1, -3, -5, -5, -1, 7, 18, 31, 44, 54, 58, 54 \ldots)$

From the foregoing it would appear that formula No. 8 has greater smoothing power than any other published formula not involving more than 21 terms.* It might, therefore, not unreasonably be expected to give rather smoother results than formula No. 6. To test the formula it was applied to the same

* Exception should, however, be made of the general formula—of maximum smoothing power—given in Dr. W. F. Sheppard's paper on "Graduation by Reduction of Mean Square of Error" *J.I.A.*, vol. xlix, p. 152).—Eds. *J.I.A.*

TABLE I. *Graduation of* O^M *Table "New" data. Values of* $10^5\Delta^3 q$.

x	Spencer's second 21-term formula	Spencer's original 21-term modified No. 8	x	Spencer's second 21-term formula	Spencer's original 21-term modified No. 8	Spencer's original 21-term modified. Graduation with reference to colog p_x, $O^{M(5)}$ as standard*
10	+2	−1	55	+5	+4	+4
1	−3	0	6	+6	+6	+6
2	0	−2	7	+8	+5	+5
3	−1	0	8	+6	+7	+8
4	−3	−1	9	+4	+6	+3
15	+1	−2	60	+2	0	+3
6	0	+2	1	0	−2	−3
7	+1	−1	2	−6	−4	−3
8	−1	+1	3	−2	0	−3
9	+2	−1	4	−2	−2	+2
20	−1	+1	65	+1	+5	+2
1	0	+1	6	+7	+3	+4
2	−1	−1	7	+8	+4	+7
3	+3	+1	8	+8	+7	+4
4	−1	+1	9	+7	+2	+2
25	+2	+1	70	+3	+13	+14
6	+2	+2	1	+5	+7	+4
7	0	−1	2	+4	0	+4
8	−2	0	3	+3	+6	+2
9	−1	−4	4	+4	+8	+11
30	−1	−1	75	+6	+2	+2
1	−2	−2	6	−3	+2	0
2	−2	0	7	−1	−10	−9
3	+2	−1	8	−1	0	0
4	+2	+4	9	+1	+5	+5
35	0	0	80	+4	−2	−1
6	+4	0	1	+9	+9	+7
7	−2	+1	2	+12	+6	+7
8	0	0	3	+9	+15	+15
9	−1	−1	4	+6	+10	+10
40	−2	−1	85	+8	+6	+7
1	0	−1	6	+5	+5	+3
2	+1	+2	7	−4	+6	+7
3	+1	0	8	+4	−1	−1
4	+4	+1	9	+2	−8	−7
45	+3	+3	90	−4	+4	+2
6	+3	+3	1	+1	−3	−2
7	+1	0	2	−8	−4	−4
8	+1	+1	3	−5	−8	−7
9	−2	−2	4	−8	−8	−10
50	−4	−2	95	−15	−12	−11
1	−3	−5	6	−12	−15	−15
2	−3	0	7	−19	−19	−19
3	0	3	8	...	...	...
4	0	+3	9	...	...	...
Totals	+35 −36	+31 −30	Totals	+148 −90	+153 −98	+150 −95

* The values of q_x below age 55 obtained by graduating with reference to colog p_x of the $O^{M(5)}$ Table as standard differ from the values first obtained by application of formula No. 8 at four ages only, namely, 50, 51, 52 and 54, the value by reference to the standard being greater by unity in the last place in each case.

data (O^M "new" data, participating assurances 1863–93), as used by Mr. Spencer (*J.I.A.*, vol. xli. p. 402 *et seq.*), the resulting values of $10^5\Delta^3 q$ being shown in Table I, in comparison with Mr. Spencer's graduation; and also with a graduation by No. 8 with reference to colog p_x of the $O^{M(5)}$ Table as standard. A comparison of actual and expected deaths is given in Table II. From the results shown in Tables I and II it cannot be claimed that formula No. 8 is perceptibly better for the graduation of colog p_x than Mr. Spencer's second 21-term formula; nor do the results with reference to colog p_x of the $O^{M(5)}$ Table as standard show any material improvement. All three graduations must be considered good. Possibly if one more figure were retained in the function to be graduated the superior smoothing power of No. 8 would be shown. In Table III are given the "corrections" when graduating with reference to colog p_x of the $O^{M(5)}$ Table as standard. As a matter of interest these have been carried back to age 20, and the values of βe^x are given so as to facilitate the calculation of the "corrections" for any other formula.

TABLE II.

O^M "New" data. Comparison of Actual and Expected Deaths.

Age-group	Actual Deaths	Spencer's Second 21-term Formula		Spencer's Original 21-term Modified No. 8		No. 8 with reference to $O^{M(5)}$ as Standard	
		Deviation	Accumulated deviation	Deviation	Accumulated deviation	Deviation	Accumulated deviation
10–14	8	0	0	0	0	Same as preceding.	
15–19	91	0	0	0	0		
20–24	743	− 18	− 18	− 19	− 19		
25–29	2,303	+ 23	+ 5	+ 30	+ 11		
30–34	1,352	− 16	− 11	− 16	− 5		
35–39	5,785	+ 33	+ 22	+ 34	+ 29		
40–44	6,714	− 94	− 72	− 103	− 74		
45–49	6,712	+ 71	− 1	+ 76	+ 2		
50–54	6,745	+ 56	+ 55	+ 58	+ 60	+ 61	+ 63
55–59	6,320	− 76	− 21	− 82	− 22	− 82	− 19
60–64	5,252	+ 9	− 12	+ 14	− 8	+ 14	− 5
65–69	4,152	− 10	− 22	− 14	− 22	− 13	− 18
70–74	2,708	+ 12	− 10	+ 10	− 12	+ 10	− 8
75–79	1,399	+ 2	− 8	− 1	− 13	− 1	− 9
80–84	509	− 7	− 15	− 7	− 20	− 7	− 16
85–89	97	+ 11	− 4	+ 11	− 9	+ 11	− 5
90–94	16	− 1	− 5	− 1	− 10	− 1	− 6
Totals	53,906	+ 217 − 222	+ 82 − 199	+ 233 − 243	+ 102 − 214	+ 236 − 212	+ 105 − 184

Table III.

Corrections to be applied to colog p_x when graduated by formula No. 8, being equal to βc^x of the $O^{M(5)}$ *Table, multiplied by* $e = -\cdot 0001728$ (*see Table* A).

x	βc^x	$(10)^7 e \,.\, \beta c^x$ written positively	x	βc^x	$(10)^7 e \,.\, \beta c^x$ written positively
20	·0003842	1	70	·0253279	44
⋮		⋮	1	·0277077	48
⋮		⋮	2	·0303110	52
33	·0009133	2	3	·0331589	57
⋮		⋮	4	·0362774	63
⋮		⋮	75	·0396826	69
39	·0015653	3	6	·0434110	75
⋮		⋮	7	·0474898	82
⋮		⋮	8	·0519517	90
43	·0022418	4	9	·0568329	98
4	·0024524	4	80	·0621727	107
45	·0026829	5	1	·0680143	118
6	·0029349	5	2	·0744046	129
7	·0032107	6	3	·0813954	141
8	·0035124	6	4	·0890430	154
9	·0038424	7	85	·0974092	168
50	·0042034	7	6	·1065614	184
1	·0045983	8	7	·1165735	201
2	·0050304	9	8	·1275264	220
3	·0055030	10	9	·1395083	241
4	·0060200	10	90	·1526160	264
55	·0065857	11	1	·1669552	288
6	·0072044	12	2	·1826417	316
7	·0078813	14	3	·1998021	345
8	·0086218	15	4	·2185748	378
9	·0094319	16	95	·2391113	413
60	·0103181	18	6	·2615773	452
1	·0112876	20	7	·2861541	494
2	·0123481	21	8	·3130401	541
3	·0135083	23	9	·3424523	592
4	·0147775	26	100	·3746278	647
65	·0161659	28	1	·4098265	708
6	·0176848	31	2	·4483323	775
7	·0193464	33	3	·4904560	847
8	·0211641	37	4	·5365375	927
9	·0231526	40	105	·5869486	1014

Obituary.

JOHN VICTOR MCLEAN, Student of the Institute, Lieutenant, 6th Battalion Royal Berkshire Regiment.

Died of Wounds 17 *July* 1916.

FREDERICK JOHN GRANT, Student of the Institute, Private, 20th Battalion Royal Fusiliers.

Killed in Action 20 *July* 1916.

PERCIVAL JAMES DAVIS, Probationer of the Institute, Private, 2nd Battalion City of London Regiment.

Died of Wounds 16 *October* 1916.

STANLEY OCTAVIUS BENJAMIN, Associate of the Institute, Bombardier, Australian Expeditionary Force.

Died of Wounds 23 *November* 1916.

GEORGE EUSTACE BURROWS, Associate of the Institute, Private, Royal Army Medical Corps.

Killed in Action — *February* 1917.

ROBERT JOHN LEDGER, Student of the Institute, 2nd Lieutenant, Royal Sussex Regiment.

Died on Service 11 *March* 1917.

AUSTYN JAMES CLAUDE FYFE, F.F.A., Associate of the Institute, Lieutenant, R.F.A.

Killed in Action 25 *March* 1917.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Inflation: in what sense it exists; how far it can be controlled.

By Professor H. S. Foxwell, M.A., F.B.A.

[An Address delivered to the Institute, on 26 March 1917.]

Mr. President and Gentlemen,—I wish in the first place to say that I regard my *role* here to-night as a very modest one. I came here by invitation of your President on the understanding that the Institute desired to discuss questions of inflation, and I am here really to make a statement, which I will try to make as methodically as possible, the object of which is simply to open a discussion. My purpose, then, is rather to clear the issues for debate, and if I add an expression of my own opinion on some of these issues it may serve perhaps to present a target for the attacks which I am sure will follow, and thus to set the ball rolling.

There is no doubt that considerable anxiety exists, an anxiety which has been expressed in very competent quarters, upon what is called inflation. I may refer, for instance, to a very able article by Mr. Oswald Falk in the *Nineteenth Century* for July 1916, in which he not obscurely hints that the international position of London itself may possibly be involved if something is not done to check this inflation. Mr. Hartley Withers in the *Economist* has continually returned to this subject. Professor Nicholson has introduced it, although his study is not completed yet, in the *Economic Journal* for last December. We are waiting for the figures upon which his statement is based, and consequently I cannot now deal with his position. I might refer also to the references which have appeared from time to

time from the pen of the distinguished City Editor of the *Morning Post*, who several times has drawn our attention to the same point. Again, Mr. Withers says in *The Economist*: "We "believe that inflation has already done much harm by raising "discontent in the country and increasing the cost of the war, "owing to the rise of prices that it has helped to cause, and that "everything should be done to check it." Well, we may admit perhaps that the rise of prices is to be deplored, though I should like to point out what is, of course, very obvious—that the rise of prices is an effective check upon consumption, and at the same time a very valuable stimulus to production, and not only to production, but, if we are thinking really of the case of this country, to supply from other countries. In fact, there is a natural harmony about the rise of prices when the rise of prices is due to scarcity. What precisely is meant when it is suggested, as Mr. Withers suggests, that this rise of prices is partly due to inflation I do not know. Mr. Withers is cautious: he says "partly" due to inflation. He specifies particularly the Treasury note issue and the advances that have been made by banks in various countries, not only the advances made by State banks, but the increase in bank deposits which represent advances to the public, in any case practically an increase of what we regard in this country as currency—bankers' money.

I think it will be convenient before going further to make a more or less formal distinction between the different senses in which the word "inflation" has been used. I make no apology for introducing what some people would call an academic discussion to an audience of this kind. First, I think we have what may be called legal or internal depreciation, depreciation as known to law; that is, loss of parity of either coins or notes or other legal tender currency in comparison with the standard unit, the standard unit in this country being the gold sovereign, not bar gold. If there were an inflation and a consequent depreciation arising from this cause, it would be measured at once by the depreciation of that element of currency in terms of the standard unit. Supposing, for instance, that element were Treasury notes, we should find them at a discount in terms of the sovereign. Further, it might, if it went far enough, result in what William Cobbett used to call two prices: we should have a double price quotation for everything—a gold price, and a currency price representing this particular element of the currency which was inflated and depreciated. I am not aware

that there has been any depreciation of this kind in Great Britain. I have been on the lookout for it incessantly, but I have not come across even an isolated case, although it is difficult to prove negatives, and possibly it is conceivable that somebody may have offered a premium on the sovereign. I cannot speak so positively for France; I have been told that in France there is a premium on gold, but I do not think there is any quotation. That may again be owing to an act of State. We do not know how far the markets are free. In Germany we know there cannot be two prices. Germany enacted a law on the 23rd November 1914 which exactly corresponds to our Lord Stanhope's Act, which was passed during the Napoleonic Wars in 1811, an Act which makes it penal to buy or sell gold coin at a higher price than its face value. That law was passed to put a stop to a bullion agitation promoted by Lord King, who had ordered his tenants to pay him either in gold or paper of the value of the gold. The Act was absolutely effective. We do not know that two prices ever existed after the passing of Lord Stanhope's Act. The Americans tried a gold Act of the same kind in 1864, and were absolutely unsuccessful, the only result being that their paper at once jumped 40 points higher discount. Lord Stanhope's Act was effective, and the German Act apparently is effective. We find no quotations of gold in Germany, and we do not find double prices: prices there are presumably paper prices.

Now perhaps I may leave that point, because we are mainly interested in our own currency. Let me pass to a second sense in which a currency may be said to be depreciated. It may be depreciated in an external sense as a whole. It may lose its parity in terms of some foreign currency. In that case this international depreciation would be shown by the exchange rate, but for my part I do not admit that the exchange rate is a proof of the depreciation of the currency. I do not think it holds, *vice versa*, that we can infer depreciation of currency from the exchange rate. For instance, the American exchange was depreciated in the most extraordinary manner in the first week of August 1914. I was told that exchange on London was absolutely sold at 7 dollars instead of 4·86 dollars. Nobody ever suggested that that argued a depreciation of the American currency in that ratio. There certainly was no inflation of the American currency. At any rate, nobody I suppose imagines that the American currency was in any way depreciated. The high exchange rate arose from circumstances mainly accidental

—the refusal on our part of accommodation which we had been usually ready to extend to America at that time; and many other cases of the same sort might be taken. I could never agree with the Bullion Report of 1810 in holding that the premium on exchange was a necessary proof of the depreciation of our currency and it is worth observing that most of the very eminent Cambists who gave evidence before that Bullion Committee were opposed to the Report of the Committee on that point. They were, perhaps, in the best position to know what accidental circumstances will sometimes disturb the exchange, and they held that it could not be taken as proof of the depreciation of our note at that time.

It is worth while, however, to see what the rates are at the present time. I will divide the exchanges into three groups: first, the more important Neutral exchanges, secondly the exchanges with our chief Allies, and thirdly the Scandinavian exchange. The most important Neutral exchange is the United States, and we know that that exchange dropped rather unfortunately in September 1915, but for the last fifteen months or so it has remained remarkably steady at 4·76½ instead of 4·866, a discount of a little more than 2 per-cent. It is difficult to know for certain whether it is a discount at all, because we do not know what the gold point is. We do know that owing to the freights and the insurance rate the range of the gold points has been very greatly widened during the war. I have tried many times to get a statement in regard to the actual gold point, but it is not forthcoming. Perhaps there is no very definite market. But I am told that practically the 2 per-cent does not represent much more than the distance between the gold point and the par; at any rate, it is a fairly steady exchange. With Amsterdam, another important exchange, at the present moment the position is fairly favourable, 2½ per-cent discount. I do not lay much stress upon the discount on the Amsterdam exchange, because it is conceivable that there are objections to settling that exchange by the export of gold, as Amsterdam is dangerously near to the Enemy; so that even if it were a discount of 10 per-cent I do not think that would prove any depreciation in our currency. It might be interesting as a sign that we were restricting the export of gold to Holland and nothing more. When we come to South American countries there is a rather unpleasant discount in those exchanges. Roughly, averaging the countries, the discount is about 7½ per-cent. With Spain the

discount is nearly $11\frac{1}{2}$ per-cent. and with Switzerland nearly 5 per-cent. The Spanish and the Swiss exchanges are, perhaps, not very important to us, but the South American is more important. It may be held, of course, that the freight and insurance are extremely high on the South American exchange, and that that explains the slump, as it were, in the rate.

When we come to our Allies our position is most favourable. However depreciated our own currency is, it is clear that if the rate of exchange is a test their currencies are in a worse position. The exchange on Paris is 10 per-cent premium, Petrograd $76\frac{1}{2}$ per-cent, and Italy nearly 48 per-cent. I do not want to insist upon that position. Then when we come to Scandinavia we find that all the Scandinavian exchanges, taking them as a whole, are $7\frac{1}{2}$ per-cent against us, but that is very easily explained. Scandinavia, at the present moment, is not on a gold basis; it is true she is using gold, but she has appreciated gold deliberately and $7\frac{1}{2}$ discount there largely represents the extent to which gold in Scandinavia is appreciated beyond the value which it has in other parts of the world. They have done that after deliberate reflection, with a view to stopping the rise of prices in their own countries; because they conceive that after the war there will be a fall of price, and they want to cut off the upper peak of the movement, to stop the rise and so to lessen the distance from which they can see there will be a fall afterwards. It may be good policy. Of course there are a great many complicated reactions in the operation of exchange rates, and it must be rather difficult to sum up the net result of this exchange policy on the whole. However that may be, it is obvious that is a remedy we never could apply here. We do not wish to keep gold out of the country—far from it. We have to make such enormous payments to New York, to say nothing of any other centre, that we want all the gold we can get. We cannot make a formal and deliberate appreciation of gold here as compared with other countries. As to the German exchange, about the only test of it—the most important test—is the New York exchange, a discount of 30·7 per-cent as compared with our 2 per-cent on New York. But it must be observed in regard to all this matter of the exchanges that the exchanges are not working freely; first because trade itself is restricted, both exports and imports, and secondly because the ordinary arrangement by which the balance of trade is adjusted by gold movement is almost completely paralyzed by the action of the belligerent Governments. Gold

cannot move freely. Mr. Withers very wittily put it some time ago: "It has been said that the world is now divided into two "classes of countries: those which refuse to receive gold and those "which refuse to part with it." I think it is clear from the rates of exchange that even Great Britain does not go out of its way to facilitate the export of gold. I will leave it at that. I do not admit with Professor Cassel, who has written a couple of articles in the *Economic Journal* on this point, that we can argue from discount on exchange that the currency of the country whose foreign exchange is at a discount is depreciated. Present circumstances are too exceptional for us to draw any inference of that kind. It might possibly be drawn in peace time.

Now I come to what I think is probably at the bottom of people's minds when they speak of inflation. They are not thinking so much of the parity between our currency and some other, or the parity between one element in our currency and our standard. They are really thinking of the rise of prices. It is that in their minds that argues inflation; at any rate, it is the rise of prices that they wish to correct. This, of course, is entirely to change the point of view. Here we are considering, not the depreciation of something in terms of gold, but the depreciation of gold itself: and that is extremely important, because it is clear that the depreciation of gold itself is not a local question, but a world-wide question: it is not a question that we can control here, even if we wish to. Gold is depreciated nearly as much in the United States as it is in Great Britain. It would be quite as much if trade were absolutely free between Great Britain and the United States. I have been unable to get a formal index number from the United States, although I ordered it more than five months ago. As far as I can judge from isolated quotations and remarks, the level of prices in the United States is about as high as here. Two days ago I had a letter from Dr. Irving Fisher, who is the greatest authority on this point, and according to a chart which he sent me I make out that prices in the United States have risen 60 per-cent since the war began. It must be remembered that they were very much higher than our prices before the war—I cannot say how much at the moment, as we have no measure of the difference—but everyone knows that prices in the United States were, on the whole, much higher than our prices. Therefore a 60 per-cent rise would probably bring them somewhere near the level of our prices now, which have risen 100 per-cent. There is probably no important difference

between the average levels in the two countries. We have no doubt what the rise has been in this country. We have accurate index numbers, more particularly the number established by Mr. Sowerby and published by the *Statist* newspaper, which I think would be correct within a limit of error of about 5 per-cent. According to that number, prices have exactly doubled since war broke out. I ought to say, however, that when the war broke out in July prices were rather lower than they had been; they had fallen from 85 in the previous three years to 81, and in the opinion of many persons we were on the eve of a crisis, or a depression at any rate, of trade—one of the ordinary periodic depressions of trade—there being a general depreciation not only in the prices of commodities but in the prices of securities. Possibly in the case of securities the war was casting its shadows before, to some extent. There had been a drop in prices in any case, so that it is perhaps hardly fair to take that line as a base when we are calculating the rise due to the war. It would make a difference of some 4 points. But, broadly speaking, prices have not risen more in this country than in other countries, certainly other European and western countries. I have been trying to make out accurate figures, but they have no index numbers in most countries, and one can only judge from estimates made from time to time: but it is quite clear that prices are higher in Germany, that they are about half as high again in Austria, and about three times as high in Hungary. The prices there are prodigious, according to the only returns I have been able to see. In the case of meat the prices are seven times as high as they were before the war. I do not know very accurately what the prices are in Scandinavia; they have been deliberately lowered now. Complaints come from all countries of the rise of prices. It must be so, because it is gold itself that has depreciated, and as all these countries are on a gold basis, or else on the basis of currency which is at a discount on gold, prices in all countries will be similarly affected, though in different degrees.

It may be asked what would have been the course of prices but for the war? Prices have been rising on the average ever since 1896, at about the rate of 2 per-cent per annum. That rise is explained by the large increase in the gold supplies; but I do not suppose that the gold supplies alone would account for much more than a 2 per-cent rise on the average. Perhaps I ought to say a word here about the theory of prices. Price is a function of two variables: it varies directly in proportion to the

supply of money of all kinds and inversely in proportion to the quantity of goods or transactions requiring to be handled by money. The more you increase the quantity of goods handled by money, money remaining the same, the more prices fall; the more you increase money, the goods remaining the same, the more prices rise. The general relation admits of simple statement, although in working it out we are often faced with considerable detail. For instance, what is money? You have to take account of bank deposits, cheques, and the various forms of purchasing power, and that is not a particularly simple thing. But in principle nothing can be clearer or plainer. Dr. Johnson put it very well. He was told that in the Island of Skye 20 eggs might be bought for a penny, whereupon he observed: "Sir, I do not "gather from this that eggs are plenty in your miserable island, "but that pence are few."

To what, then, is due this rise of prices that has undoubtedly taken place since the war? In the first place, to the large production of gold which is the basis of all our credit, but mainly to the enormous increase of purchasing power which has been created by the various belligerent Governments, quite apart from the form which that purchasing power has taken. Our own expenditure is about £6,000,000 a day, and I think the total expenditure may be estimated at about £20,000,000 a day for all belligerents; at any rate it is an enormous sum. In our case the additional expenditure of Government is more than as large as the total expenditure of the nation in peace time. No one proposes to restrict that expenditure of Government, I suppose. The various belligerent Governments are struggling to obtain the military material and other necessaries for carrying on the war, and it would be absurd to attempt to restrict their power of obtaining that material. I do not think it makes very much difference by what precise machinery they exercise that power. For instance, if the belligerent Governments were able to buy merely in exchange for their own scrip, their own credit, without using currency at all, I do not think we should have a much smaller rise of prices. They would be still in the market for the same amount, and would be competing with the same intensity. Is there any reason for supposing, therefore, that the level of prices would be much lower? My own belief is that there is not. The level of prices is really the result of the enormous expenditure of the Governments, and as long as that expenditure is maintained and exerted the level of price will roughly be maintained.

But I do think it makes some difference by what machinery the purchases are made. In the act of purchase the effect will be the same, but if you make the purchase by means of an increase of currency you leave behind after the purchase the purchasing power you have created, and I think that constitutes a real difference in the position. You are left with a large mass of purchasing power in the hands of the general public which would not have been in their hands in that form but for the particular way in which the Government made its purchases. It would be better, for instance, as far as that is concerned, if the Government could make all its purchases by forced loans without using currency at all. By what means have the purchases been actually made? They have been made, in the first place, taking the world broadly, by enormously increased issues of notes. The French note issue was trebled and the Russian note issue increased four times. Our own note issue has only increased slightly, and the issue of bank notes hardly at all. The new currency note issue has altogether reached the amount of more than £140,000,000; but against that must be set the value of the gold whose place has been taken by the currency note, most of which gold has been exported to the United States. I think it will be found that the excess issue of currency notes over the gold displaced by the currency notes is, after all, a very small matter, not in any way comparable with the rise of prices, nor an increase that would have contributed very much to that rise of prices. Then in certain countries enormous advances have been made by the State banks. The Bank of France advanced £400,000,000 to the Government, most of it gratuitously and none of it at more than 1 per cent. It behaved in the way it always does behave in times of crisis, admirably. The State is under very great obligation to that bank, so much so that meetings have been held in France of merchants and financiers calling on the Government to extend the bank monopoly for 30 years without asking for any concessions from the bank. The extension of the monopoly is usually made an occasion for asking for concessions, but on this occasion the feeling is that the State owes so much to the Bank of France that the bank has earned its monopoly for another 30 years without further consideration.

Then in all countries there has been a very great increase of divisional money, silver and other forms of small change. Perhaps that does not count for very much, but I think it has some effect. In this country in normal times the circulation of silver

is about one-fourth in value of the circulation of gold, so that it really is a considerable element in our price basis. What is most important for this country—but not so important I think for many others—is the increase in bank deposits. We are accustomed to consider the draft upon a bank as the very best form of money, cash in the highest sense of the word, especially if that bank happens to be the Bank of England. A draft on the Bank of England would be considered cash, perhaps, in any part of the civilized world. In this country we make a very large use of Bank of England advances when we are in difficulties for want of currency. I have found very great difficulty in estimating what has been the precise increase of bank deposits in this country. Our banking returns are always miserably inadequate and unworthy of the country: but during the war, ever since the large loan of 1915, they have been suspended altogether, so that we only have the annual account made up for a particular day, and everyone knows what that means as a basis for an estimate. We have, in fact, no basis at all. I think it is a matter to be very much deplored. You will remember what was said about Austria when the Austrian Bank suspended its returns. I am sorry that we should have in any degree followed an example of that kind, because I think myself there was no reason whatever for it. I cannot see that the banks have anything to be ashamed of. It is a sort of morbid timidity that seems to cause the withholding of these returns without which there can be no proper basis for scientific action. However, as far as I can make out, the bank deposits have not increased more than some £250,000,000 during the war, to which may be added about £120,000,000 for the Bank of England, or under £400,000,000 altogether for the whole banking system.

Compare that with the total of deposits before the war of something like £1,200,000,000. There is a 33 per-cent increase of bank deposits, and that is the most serious figure relating to inflation that can be found in our accounts—the expansion of bank deposits by something like 33 per-cent. But that will not account for 100 per-cent rise in prices, nor is it anything like so large as the correlative forms of expansion in other countries, the expansion of notes and State bank advances and so forth. The State bank advances have been very small here as far as one knows. I do not know what the advances may be at any given time, but supposing them to be £70,000,000 or £80,000,000, we have no figure that can be classed as a figure of expansion here that

at all corresponds to the great movement of prices. It must be observed that in some foreign countries they have been increasing the use of cheques, and that has been another cause of rise of prices. In France, Russia and Germany the greatest efforts have been made to extend the use of the cheque, and that has had just the same effect as the expansion of the note issue. Generally speaking, I am inclined to think there has been an economy in the use of metallic money and in the use of notes all over the world—that there has been a more rapid circulation of money. During the Napoleonic wars it was proved that the rapidity of circulation of the bank note doubled during some ten years. I admit the case is not quite parallel. The fact was that we were just beginning to understand banking economy at that time; our clearing house had been established only twenty years before, and we had just begun new experiments in the machinery of cheque banking, and it is possible, therefore, that the increased velocity was not due to the war but to the natural development of our banking system. But I am inclined to think that the pressure caused by war does make the sixpence a little more nimble than it otherwise would be. Then there is the question of the emptying of hoards. Large amounts have been brought from hoards in France and in Germany. I do not know if we shall get any in India from the new Indian Loan, from the great hoards there, or through ornaments being melted down. This has been an addition to the amount of gold available for currency, and just as important, in its effect on prices, as if new gold had come from the mines. These are the general causes, on the money side, of the rise of prices.

Passing to the side of commodities, there has been a shortage in productive power tending so far to make commodities scarce, but of course this has been compensated very largely by increased effort and by women's labour, and by the labour of other persons not usually employed in normal times. I am bound to say there has been in this country a great margin for an increase in productivity. When you hear that women coming fresh to an industry were able to turn out five times the previous output of a skilled workman, it is impossible not to feel that we were working well within our power when the war broke out. I would also refer to the decrease of available tonnage, and the rise of freights. High freight rates do not, strictly speaking, I think, restrict the supply of commodities generally, but they may restrict it locally. They may prevent us, for instance, from

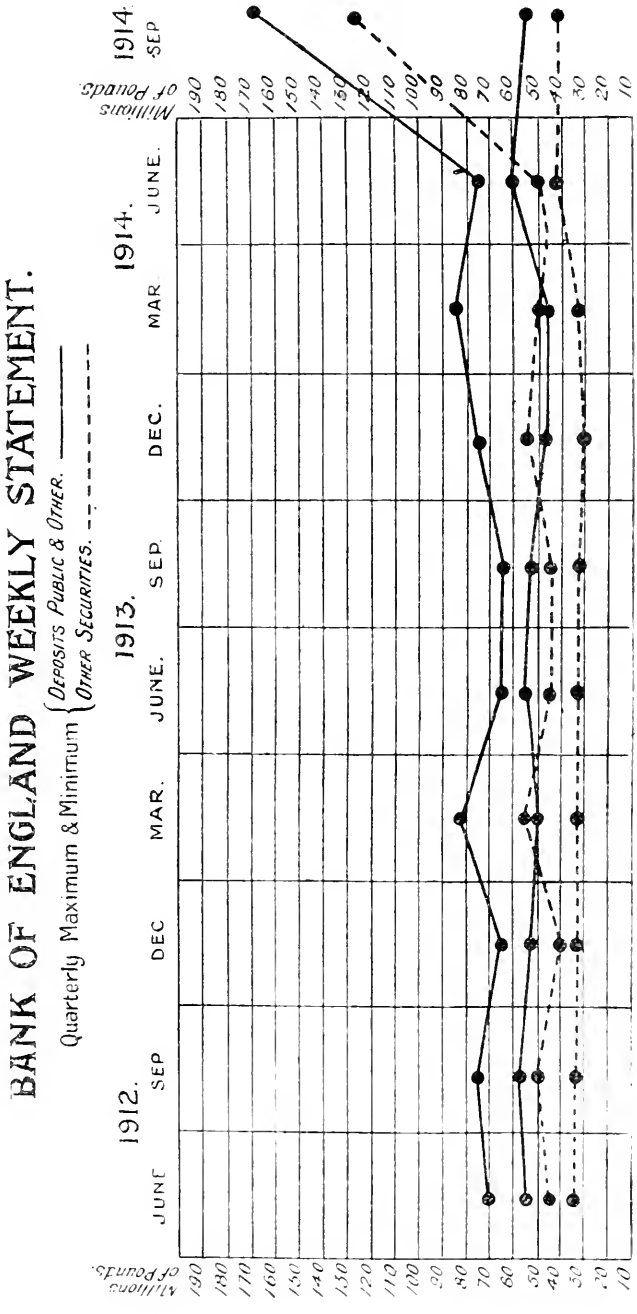
bringing goods from the United States which we might otherwise have had, and they may bring about a local scarcity here and therefore cause local high prices. There has been a certain failure in the harvest, partly due to the shortage of fertilisers. Those, I think, are the principal causes on the side of commodities.

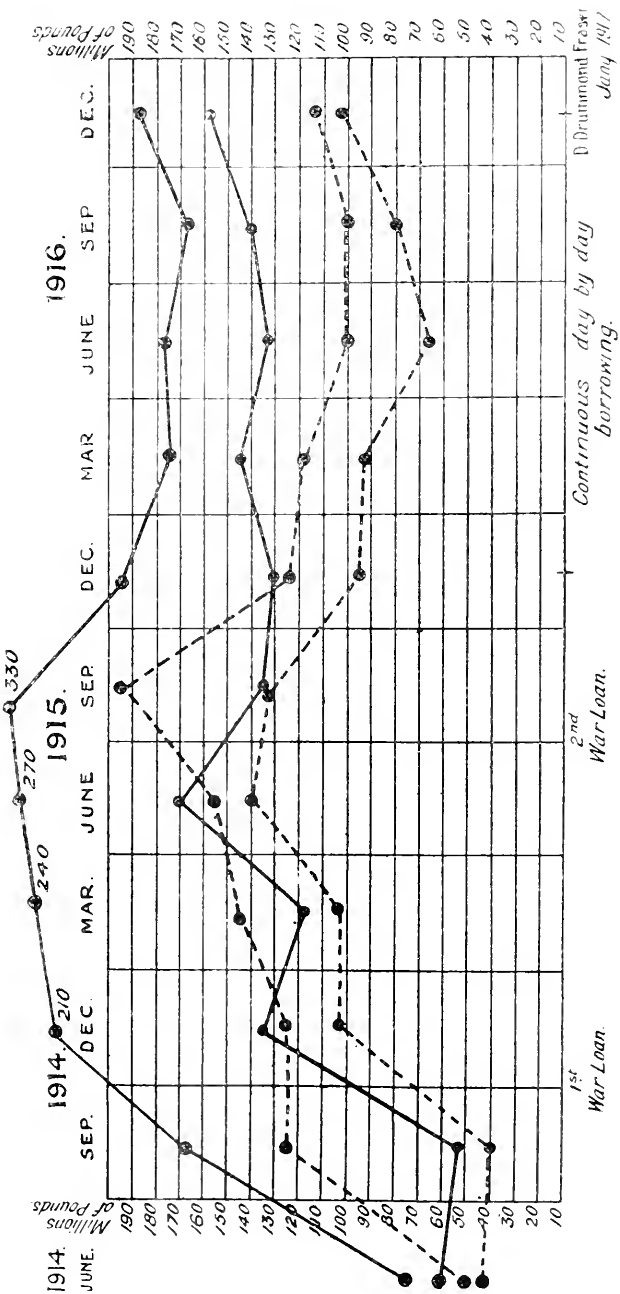
Then there are certain matters connected with dealing and distribution which have tended to raise prices. There has been too much of the amateur in the market. He generally pays very dear for his operations. Military purchases, for instance, have not been of the most economical type. On the other hand, there has been a certain tendency on the part of the expert to hold up goods and to exact the full advantage of his position and his knowledge, and there has also been a certain amount of hoarding by consumers. These are small matters, and not so important as the others I have noticed.

I ought, perhaps, to say a word about the question of bank deposits in addition to what I have already said. I said that the way in which Government borrowed did make some difference, and it was advisable, as far as possible, to avoid borrowing by the creation of currency. We have done very little to create currency in this country in the shape of notes, practically nothing if you make allowance for the substitution of notes for sovereigns; but we have done a great deal—say to the amount of £400,000,000—to create bank currency by bank advances. The question is whether that could in any way be avoided, and I am inclined to think that it can be avoided, that it is very largely connected with the issue of big loans. These big loans require large operations to finance them. We talk about raising a loan of £1,000,000,000, but it is quite certain that there does not exist in the country at any one time even £400,000,000 of spare cash, and it is an impossibility to raise a loan of £1,000,000,000 in the strict sense of the word. What we do is to finance it by bank advances. Every big loan means a large expansion of bank deposits. We cannot tell how much because we cannot get the returns, but we know very well there must be that expansion, because we know there is no spare money in the country to the extent of anything like the sum raised by a big loan. No country in the world could raise a loan, I think, of £500,000,000 without recourse to some method of financing, and that method of financing practically creates currency. It is for that reason that I confess myself entirely a partisan of Mr. Drummond Fraser, the apostle of continuous borrowing in this country, who

argues that the proper way to borrow is to take up money when the public have it to spare by always being open to receive loans, to receive the spare cash as it comes into the current accounts. That is practically what we were doing all through 1916. I admit that there were reasons for one large regularising loan. We had got into difficulties with previous loans and previous rights conveyed by those loans, and it was desirable to unify the public credit on a single basis as far as possible, creating a large marketable stock on sound principles. That was admirably done by the recent loan. It is likely to be what Consols used to be, a fine banking security in the future, or at least I hope so, and I was very glad to see that it was so admirably insured by the large contributions made first by War Saving Certificates and secondly by the Prudential Insurance Company and other bodies of that particular type. It seemed to me a very important point, a bull point for the new stock. In the future and in general it seems to me that Government can raise the money it requires with a minimum of disturbance and with a minimum of inflation if it will avoid placing these large loans at long intervals and revert to the system of continuous borrowing to the full extent as money becomes available. That borrowing should give a freedom of option to the public; there should be bonds of various dates and bills of various dates. The chart on pp. 280-1 constructed by Mr. Drummond Fraser shows the effect upon the deposits of the Bank of England of two kinds of borrowing during the war. The upper part of the chart shows the pre-war period, the normal position of the Bank of England deposits as shown by the Returns. When you get into the war period the deposits rise very rapidly, partly owing to the panic at the outset of the war and afterwards owing to the War Loan; but when in 1916 you come down upon this method of continuous borrowing you see how remarkably even the curve of Bank deposits is, almost at a level from December to December. I have no doubt that if we could get the figures for the new Loan we should see a corresponding rise in the bank deposits, if not of the Bank of England yet of the country as a whole, in consequence of the big loan of 1917. It is well to avoid these disturbances, and also to avoid the creation of currency which they cause.

The root of the whole matter it seems to me is this: whether by inflation or otherwise it is gold itself that has depreciated, or, in other words, it is the depreciation of gold which has caused the high level of prices. This depreciation of gold is the result


BANK OF ENGLAND WEEKLY STATEMENT.
Quarterly Maximum & Minimum
Deposits Public & Other.
Other Securities.
1912.
1913.
1914.
1914
JUNE
SEP
DEC
MAR.
JUNE.
SEP.
DEC.
MAR.
JUNE.
SEP
Millions of Pounds.
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10


Millions of Pounds
1914. JUNE.
SEP.
1914. DEC.
MAR.
JUNE
1915.
SEP.
DEC.
MAR
JUNE
1916.
SEP
DEC.
210
240
270
330
1st War Loan.
2nd War Loan.
Continuous day by day borrowing.
D Drummond Fraser Jany 1917

of the enormous increase of purchasing power in the hands of Governments. In some cases the new purchasing power has been created by methods which have depreciated currencies in relation to gold. That does not seem to have been the case here. Our prices are very high. But we see these high prices in America and other countries where there is no question of inflation in any ordinary sense of the term. Thus the problem is international, and that is a matter of the first importance, because it shows that, unless we are prepared to revert to what I may call Scandinavian methods to bring about a local appreciation of gold in our own country—and that is obviously impossible and undesirable—the rise of prices is beyond the control of any one country. We can only check it by cutting ourselves off from a gold standard, which is what we do not want to do, either by appreciation or depreciation. Secondly, I think it is important because it meets an objection raised by Mr. Falk which I confess impressed me rather at the time I read it—that a rise of prices by creating exchange difficulties threatened our international position after the war. It might if it were peculiar to this country, or even if it were peculiar to belligerent countries; but if it is really universal, if it affects the United States, for instance, I cannot see how it threatens our international position. At any rate that consideration reduces the danger to which Mr. Falk referred. That is broadly the position that I wish to submit.

I have here the figures which I did not give you just now. The war expenditure of the five principal belligerents is £18,500,000 a day, and if you add the others it will not be far under £20,000,000 for the whole of the belligerent Powers; say £7,300,000,000 a year, or, if we allow for double entry in connection with loans and so on, roughly £7,000,000,000 a year war expenditure. This is the root fact, I think, at the bottom of the inflation.

Abstract of the Discussion.

Mr. O. T. Falk said the members had listened to an exposition of an extremely difficult subject by a master mind. The address was based on the study of a life-time, and the subject was not one usually discussed at the Institute. He asked the members, however, to listen to his own amateurish remarks for a few minutes mainly because he hoped that by taking part in the discussion he might extract a little more truth from Professor Foxwell. Professor Foxwell had said that the objectors to inflation were mainly concerned with the third form of inflation in his category, the depreciation of gold.

That might be so, but it was not his own experience, possibly because his life was spent in the City, and he thought it would be admitted that, so far as those who lived in the City were concerned, inflation of the first two kinds was the most important. It might be because he had not a due allowance of that morbid timidity which Professor Foxwell ascribed to bankers that he disagreed with him with regard to the position in so far as the first two forms of inflation were concerned. There was a difficulty about the question and he frankly admitted that, so far as he knew, those who believed that depreciation —or what he would rather call potential depreciation—of the currency in terms of gold existed, were at present unable to offer any satisfactory proof of that depreciation. So far as he was concerned, he suspected it. He did not admit that the rise of prices in every country was a measure of the depreciation of gold, simply because he did not admit that the prices in all those countries were gold prices. He believed they were admittedly not so in Russia, to take a single example, and he thought they were not so in other countries also.

He appreciated Professor Foxwell's point when he said that there was no cause for alarm with regard to the position of this country if the rise in gold prices was international and if the rise in this country was not greater than the rise in other countries: but his own idea was that the rise in prices in most countries was a rise in currency prices and the rise was measured in currencies which were admittedly depreciated in terms of gold. In the case of this country, the currency was supposed to be on a par with gold. If, therefore, our prices had risen to as great an extent as the prices of other countries, it was possible that our currency was depreciated in terms of gold; in other words, it was a question as to whether there was not what might be strictly called relative redundancy of the currency. It was not possible for him to give any proof of that relative redundancy, but he would ask the members for a moment to assume that it existed. He wanted them to assume it because he wanted them to allow him to criticize the point of view of those who said that redundancy simply could not exist, who said that the absence of that redundancy was proved by certain current conditions. Professor Foxwell had compared the position in 1810 with that of 1917, and it would appear that that was a very good starting point for the criticism he himself wished to make. In 1810 the Bullion Committee and Ricardo and a few others pointed out that the high price of bullion and the state of the foreign exchanges was a proof of the depreciation of bank paper, and a proof also of the over-issue of that paper. He would agree with Professor Foxwell that the depreciation of the exchanges was not a conclusive proof in all circumstances of the depreciation of the currency in terms of gold, but for his purposes that qualification was not important, because the point he wished to make was that, whether those tests were conclusive or not, it was fair to say that it was not possible to state with confidence that, because there was no high price of bullion and because there was no depreciation of the exchanges, therefore redundancy was absent.

Comparing the position in 1810 and the position to-day, in 1810

there was a high price of bullion and there was a Restriction Act. The objectors to the hypothesis which he put forward said that to-day there was no Restriction Act and no high price of bullion. He admitted there was no Restriction Act, but there was restriction by consent. It was also said that there was no high price of bullion, but he did not see how there could be a high price of bullion; it was not possible to melt down gold coin or to import bullion or to export it—as a matter of fact bullion could not really be bought. There was no Lord Stanhope Act, but it was not legal to pay a premium for British gold coin, and certain men had been convicted of that offence. So far as he knew, there was only one case in which a gold premium within the country could be made evident, and it so happened —although he laid no stress upon the point—that in that case a premium existed. It was legal to purchase foreign coins, not current coins, and melt them down, and jewellers, because they badly needed gold, were buying foreign coins and paying a high premium for them, from 10 to 15 per-cent, but it was not a highly regularized trade, so that the premium probably varied considerably in different localities. He did not in any way wish to suggest that the premium paid by those jewellers was any evidence of a gold premium within the country, but he did think that it was fair to say that the absence of a gold premium was not a proof that there was not a premium on gold, because by consent and by legal restriction there was no real dealing in bullion or gold coin. Fortunately this country had been free during the war from individuals of the type of Lord King, so that we had not really had the point tested as it might have been.

The second main test of the days of the Bullion Committee was the state of the foreign exchanges. The American exchange was now at the low gold point, or approximately so, and he was perfectly prepared to admit that the state of every other Neutral exchange could be explained away. He did not want to go in detail into the matter, because he thought it must be obvious that if this country had been able to maintain the American exchange at the low gold point it could have maintained all the other Neutral exchanges at that point by the same methods, if there had not been special difficulties. Assuming that all the foreign exchanges were at the low gold point, could it therefore be argued that the currency was not potentially depreciated? He thought not. The state of the exchanges in the present case proved absolutely nothing. He would ask the members to look for a moment at the method by which this country was maintaining the American exchange: we were not paying for goods with goods; we were paying for the adverse balance of trade with gold, with foreign securities that were acceptable and with promises to pay gold. Promises to pay gold were not gold, but promises to pay gold, and that was a point that it was impossible to escape from. He thought it was fair to say—to borrow a phrase from Professor Jevons—that this country was engaged in selling a gigantic bear of gold. Anyone who knew what the selling of a bear might result in, especially when it was on a very large scale, would appreciate what he felt about the situation. If his hypothesis was

correct, this country was selling gold for future delivery at a price which might be far below the world's market price, if, as he supposed, our currency was relatively redundant. He did not say we were selling below the market price, but he thought it was possible. It was evident in any case that we could, so long as we had credit and acceptable foreign securities, go on maintaining the exchanges independently of the state of our currency. It might be redundant and yet things might for a time be kept right.

Personally, therefore, he thought there was some reason for careful enquiry into the situation and possibly for more prevision than had as yet been exercised. That necessity became obvious if one looked ahead and saw what would happen on his hypothesis, namely, if this country was selling a bear of gold with a redundant currency. Some day we should have to stop settling an adverse balance of trade by promises to pay gold, and we should then have to buy goods with goods, and if we wished to buy goods with goods we should be at an extraordinary disadvantage if we offered gold at a lower price than the rest of the world. Gold would then be the cheapest commodity that we had to offer, and the foreigners would take our gold first and our commodities second. He did not mean to say that gold would be the only thing they would take from us, but gold would be very nearly the first, if not the first, they would take from us. The question was—How was the problem to be solved? It could only be done in one of three ways, unless we wished our last ounce of gold to be drained from the country. We could lower our prices by contracting the currency, which would be a highly objectionable course—one had only to read Professor Foxwell's admirable preface to the translation of Andreades' "History of the Bank of England" to see what his view about that method would be—or we could suspend specie payments, and, as the financial centre of the world, that was a solution we did not want to have to face. These two solutions were within our control. The third was not within our control, and personally he thought it would be madness to count upon it. If one could suppose that after the war some of the main commodities which this country produced and exported would rise in value relatively to the other main commodities of the world our position would be maintained, because relative prices must be adjusted to relative values, and we should, by the appreciation in value of our main commodities, be able to support a higher level of prices. There might be a fallacy there, but he thought there was not.

He did not think there was any disadvantage in discussing the position, because there was very little harm that could be done under present circumstances and he was not sure there was not a great deal of good, but he did wish to emphasize very strongly that he by no means maintained that a relative redundancy existed; he admitted he only suspected it. The remarks he had made were mainly a plea for a very careful enquiry, the nature of which there was not time to develop, but which had really been indicated by Professor Foxwell. He was hoping that those who had the capacity for undertaking

that work might combine and test the position, so far as it was possible to do so, by obtaining the necessary information from other countries. He understood, from the article to which Professor Foxwell referred, written by Professor Nicholson in the last number of the *Economic Journal*, that Professor Nicholson was engaged on such an enquiry, and he trusted that he would get all the support that was necessary. Also he hoped that it would be possible to induce the City to take the matter rather more seriously than they appeared to do at the moment. The City referred the question back to the tests of the Bullion Committee days and appeared to have satisfied itself that there was nothing to be alarmed about.

Mr. W. A. KIDDY (Financial Editor of the *Morning Post* and Editor of the *Banker's Magazine*), speaking on the invitation of the President, said that he would like to express his appreciation of the address that Professor Foxwell had given. When he heard Professor Foxwell speak he always felt that he enjoyed the great advantage of listening to a deep student of economic subjects and at the same time to one who seemed to think it worth while to keep himself up to date in practical affairs. Many of his remarks that evening showed that he was manifestly in touch with the very latest development in finance. One thing that must have struck everybody that evening was that the subject under consideration was one where proof seemed to be well-nigh impossible; that had come out in what Professor Foxwell had said and also in the remarks of Mr. Falk. The question of inflation had to be considered from three standpoints: First, whether there was a premium on gold; secondly, the position of the exchanges; and, thirdly, the rise in the price of commodities. In all those three cases it was easy to find simple explanations of the phenomena, without including inflation. In considering the question of a premium on gold, Mr. Falk had pointed out quite properly that, while it could not be proved there was a premium on gold, it would be equally difficult under the artificial conditions which existed to say that there was *not* a premium. With regard to foreign exchanges, as was well known, there was a great trade balance against all the belligerent countries which were buying from the neutral countries, at a time when their own productive power was reduced. Under those circumstances, how was it possible to have anything but adverse exchanges? Therefore there was an ample explanation of the position of the exchanges without going into the question of inflation.

With regard to the rise in commodities, he had been rather struck in looking over the record of prices to see how a rise commenced about the year 1906. When the war period began there was of course a tremendous bound, but there was quite a substantial rise between 1906 and 1912, and he suggested that one reason for that rise was probably to be found in the greater equalization of wealth, the manner in which taxation was changed during those years having caused wealth to be spread over a greater number of the community: in other words, he believed that in those years there was a tremendous increase in the purchasing power of the people, and that fact was

important in connection with the more recent events, because the rise in wages, the expenditure of belligerents, and the high wages paid to munition workers, played a great part in the rise that had taken place in commodities. Wages had been high all over the world and the purchasing power of the people had increased, and under those conditions there was bound to be a great rise in commodities, quite apart from such circumstances as the cutting off of productive power and the great demands on the part of the belligerent Governments. Moreover, in the years from 1906 to the period of the war there was reason to suppose that in many of the food-producing countries there had been a tendency to go into manufactures rather than to increase the area of food stuffs; in other words, the cultivation of food stuffs had not kept pace with the growth in population. He only pointed that out because it seemed that from 1906 to 1912 there was a great rise in the price of commodities when nobody talked about inflation at all, and when there were other causes to account for it. All the same he thought there was inflation to-day, that there *must* be inflation. It seemed to him that there was no need to look further for proof than the fact that, as Professor Foxwell had pointed out, the Continental banks of Russia and France had increased their note circulation enormously. How could anything but inflation result from that? The point was that it could not be measured because it was so mixed up with the other and more easily discernible causes, and no measure could probably be found until after the war, when some of the other causes had ceased to operate.

With regard to the currency notes of this country, he was inclined to think there was perhaps rather more concealed inflation than Professor Foxwell acknowledged. It was well known that against the 140 millions of Treasury notes there was $28\frac{1}{2}$ millions of gold earmarked, and to that extent there was of course no inflation. He understood Professor Foxwell to suggest that the great mass of the remaining Treasury notes might not be inflation because of the gold it displaced, but he went on to say that a great deal of that gold had now been exported to the United States. If that were so, was not that gold forming a basis for fresh credits which, in considering the rise of prices of commodities all over the world, must lead to a still further basis of credit and therefore to inflation? He thought that the extent of the Treasury note issue was undoubtedly another sign of inflation. Was there anything in all this that could be controlled during the war? When it was remembered that the greatest cause of all was the great enlargement of credit, it would be seen that the problem was a delicate and difficult one, because we knew very well the whole war had been carried on by a great expansion of credit. If that credit were to contract suddenly, a totally different condition of things might be brought about, and yet in the enlargement of trade in various forms, Government loans, banking deposits and so forth (inasmuch as increased purchasing power was involved) was to be found one of the chief explanations of such a phenomenon as the rise in the price of commodities. With regard to the question of control, one method concerned the Government and the other

ourselves. It was admitted that the belligerent Governments were now the largest buyers of commodities of every kind, and the question was whether those purchases were being conducted on the best lines, even after making all allowances for the fact that things had to be done in a hurry. If they were not, but were being done in an amateurish fashion, then the evil was unnecessarily exaggerated. With regard to the public, their duty consisted in economizing in the matter of consumption. Although the war had been going on for two-and-a-half years, it was only within the last few months that the question of personal economy had been really forced home upon the people by the Government. If the people had not economized it was not altogether their own fault, the word not having been clearly given by the Government itself until within quite recent months.

When the question of controlling the position was really considered, however, it must be felt how difficult was the task. It was, perhaps, as regards the form of borrowing and the manner of regulating the expenditure that opportunities were chiefly afforded. With regard to borrowing, Professor Foxwell advocated very strongly Mr. Drummond Fraser's plan for continuous borrowing by Treasury bills, Exchequer bonds, etc., rather than a big loan operation, but had the Professor considered whether that form of borrowing could sufficiently draw money from the investor himself? Because, unless the money was obtained from the investor, and not simply from banking credits, Professor Foxwell would be the first to say that there would be undue credit expansion. It was easy, of course, to "job backwards," as was sometimes said in the City, but looking at the great success of the new Loan it seemed almost a pity that there could not have been some scheme devised whereby the Loan could have been made very much larger with payments made by fortnightly instalments over a period, say, of two years, the Government having the right at any time to stop the subscriptions if the war were suddenly to cease. Spreading the payments over a more lengthy period would have given the Government probably two or three times the amount they actually got, and the public, having been committed to supplying the Government with funds for perhaps two years, would have felt all the greater need for economy in the matter of personal expenditure.

Mr. E. W. TOWNLEY wished to make a few remarks on Professor Foxwell's third interpretation of inflation, namely, the depreciation of gold in terms of commodities as measured by the rise in prices. The real and ultimate cause of the rise, as had been said that evening, had been the enormous additional expenditure of the belligerent Governments, which had affected neutrals as well as themselves, but the difficulties began when an attempt was made to ascertain to what extent any one of the more immediate causes was responsible for the rise. Professor Foxwell had remarked that even assuming that British prices depended solely on British currency policy, the expansion in our currency was quite inadequate to explain the rise, and he thought everyone would agree with that. But while our currency expansion was obviously insufficient in itself to account for

the rise, and allowances must be made for other factors, such as the lack of tonnage and the withdrawal of millions of men from productive employment, it still seemed to him that the increase in our currency issues had had a material effect on the rise in prices. He would like in that connection to refer to the statistical enquiry which Professor Shield Nicholson was conducting. Although that enquiry was not yet finished, Professor Nicholson had already indicated some of the first approximate results in the following terms:—"There has been a general conformity between the increases of our note issues and the rise in prices in the United Kingdom. The increase in prices as shown by the index numbers has *followed* the increases of notes, and in general, the movement in prices is of the character associated with over-issues of inconvertible paper, such as has occurred in the other belligerent countries." As a matter of curiosity he was led by Professor Nicholson's remarks to compare the growth of our currency note issue with the rise in prices as shown by the index numbers of the *Statist* and *Economist*, and for that purpose he divided the period since the commencement of the war into intervals of three months. He did not overlook the fact that our banking deposits were so much potential currency, but, as Professor Foxwell had pointed out, the miserable monthly returns formerly published by the Clearing Banks were discontinued in June, 1915, and it was not, therefore, possible to trace the growth of their deposits at short intervals. His rough analysis, on which too much should not, of course, be built, indicated two rather interesting features. First, each of the two quarters in which the emission of Treasury notes was the largest was followed by a quarter in which the rise in prices also was specially marked. Secondly, and conversely, each of the three quarters in which the notes issued remained more or less stationary was followed by a quarter in which prices also were practically stationary, or even receded. In other words, there seemed to be a curious correspondence between the movement of currency notes in one quarter and the course of commodity prices in the next. Allowing for the time which it took currency notes to get into active circulation, that did seem to support Professor Nicholson's statement that the rise in prices had been due to a considerable extent to currency issues. If that view was correct, there was at all events one factor which it was within the power of the Government to control, and as so many other factors which were contributing to the rise in prices were beyond control, he thought everyone would agree that the Government ought as a matter of duty to restrict currency issues as far and as quickly as possible.

The PRESIDENT, in moving a vote of thanks to Professor Foxwell, said that, whatever differences of opinion there might be about the questions discussed that evening, everyone would be united in thinking that it was their duty to pass a very hearty vote of thanks to Professor Foxwell. It had been a great personal pleasure to him, and he was sure to all those who were associated with the enterprise, to have succeeded in bringing Professor Foxwell there. The present state of national affairs had, owing to so many of the members being away, acted prejudically on the power of the Institute

to hold its usual meetings, and the situation had been one of great difficulty, but still he did not think there was any need to look upon it as an unmixed evil, in so far as it had given the Institute an opportunity—which in other circumstances it might have been difficult to obtain—of getting together two such meetings as had been held during the present session. There was nothing that did the members of the Institute greater good than to see and hear and confer with and exchange opinions with men who, while not interested in their profession on its technical side, had, as the result of years of study and experience, made themselves familiar with some of those great financial problems with which the actuarial profession, like all financial interests in the country, was intimately concerned. The immediate business of the actuary took up so much of his time that he was not always able to get that grip of financial questions that he might desire, and therefore it could only be beneficial to have such opportunities as the Council had been able to provide during the present winter.

Personally he could hardly say how much he had enjoyed the address that evening, and how very strongly he felt what Mr. Falk so happily put—that the opinions expressed and the remarks made by Professor Foxwell, in his charmingly unassuming and conversational way, were the result of the study of a lifetime. One felt one was listening to someone who had been familiar with the questions he was speaking about day by day and year by year, who had studied them in all their aspects, and who, even at a time of such unprecedented difficulty as this, could give what were really counsels of wisdom. Whether one agreed with all Professor Foxwell said or not, it could not fail to be suggestive, helpful and illuminating.

On the question itself he preferred to say nothing: first, because the hour was so late, and secondly because the subject was surrounded by such enormous difficulties. There seemed to be so many possible causes, so many plausible explanations, for almost every one of the financial phenomena which were now exhibiting themselves, that having regard to the fact that we were living in a time without precedent in history, to form anything like reliable conclusions or useful forecasts was a matter of extreme difficulty, certainly not to be attempted by anyone who had not had a great deal more intimate experience of the kind of financial problems under discussion than fell to the lot of many actuaries. All he could say was that what he had heard that evening had been extremely suggestive and helpful. enabling him to realize the relations of some things which hitherto might have seemed unrelated; helpful also in so far as it impressed upon him the old lesson which had been brought home by every speaker, and very emphatically by Mr. Kiddy, of personal economy in the public interest, personal conduct of the affairs of each one of us as members of a nation that was passing through a crisis which it had never before had to face. As we were "members one of another", it was our duty to the State so to order our lives that there might be no extravagance, no expenditure which we could not strictly justify to our own consciences.

Mr. Kiddy had thought it a great pity the Government did not remind us of that long ago. He did not know that our consciences were going to acquit us of responsibility on that account. Suppose, for argument's sake, it were true it was only three months ago that the Government had said that in a crisis like this we ought to avoid extravagance. They had not yet told us that we ought to abstain from stealing, and many other objectionable things. After all should it be necessary? We were each provided with a conscience, and if we never acted upon it until receipt of Government instructions he was afraid some day, if there was any final account, it would be a poor excuse to say that we did not economise because the Government did not tell us to.

The vote of thanks was accorded to Professor Foxwell with acclamation.

Professor FOXWELL, in reply, said that in the first place he would like to thank the members of the Institute for the very kind way in which they had spoken of the few remarks he had made. The President had rather hinted that although he had brought forward very little definite evidence in regard to any statement he had made, the statements were not made at random but really represented the result of a considerable amount of thought from time to time. He could assure them that he had spent two or three weeks in trying to get some more definite evidence, but had to abandon the attempt, as he found he had no evidence worthy to present to a body like the Institute of Actuaries that would really stand the test of close examination. There were no adequate returns. That must be his apology for what he was sure must have been noticed to be a rather obvious absence in a statement such as he had made—the lack of a definite statistical basis. It was not for want of taking pains.

With regard to the admirable criticisms that had been made, in the first place he might say that in questioning the existence of any serious inflation he had had in his mind all through such inflation as it was within the power of this country to control. What Mr. Falk had said really related to the rise of prices, due to general inflation all over the world, very little of which this country could control; and he did not indicate any particular measure of control which in his opinion it would have been wise for the Government to adopt. In speaking of the exchanges, Mr. Falk said very truly that many of the other currencies whose exchange rates were quoted were not gold currencies. That he believed was true of Germany and Austria and Russia, but it was not true of Scandinavia, which had even a super-gold currency, if he might so term it. It would not be true of the United States, which was the principal currency with which we had relations and with which our relations were on a very fair basis. As to the case of a premium on gold in this country, Mr. Falk did not press the point, indeed he appeared to admit that the position of the goldsmith was very peculiar. The goldsmith was not buying gold as currency, but as a metal which he happened to be able just now to sell at exceptionally advantageous rates and therefore he did not care what he paid for it. On the other hand, the Government was

interested in stopping the consumption of gold by the goldsmith, and had imposed certain restrictions on his usual source of supply, and that made him all the more anxious to obtain it and tended to cause the premium. Personally, therefore, he did not attach any importance to a premium if only paid by goldsmiths; the very same thing happened in the Napoleonic wars: one of the few definite purchases at a premium he had seen mentioned was a purchase by a goldsmith and there was one conviction, quashed on appeal, for a sale of guineas at a premium for export. Then Mr. Falk said there was no real dealing in bullion now. Probably that was so. In any case, even in time of peace, the bullion market was very limited, confined, he believed, to about four firms. He did not understand that those firms had absolutely nothing to do at the present moment, but he could easily believe that the dealing might be very restricted. The rate quoted for bar gold was the lowest possible, namely, £3 17s. 9d., the Bank price.

Mr. Falk rather deplored the method by which we were maintaining the exchange. It was in fact deplorable that the exchange had to be maintained by exporting securities, but the export of securities was just as legitimate as the export of goods. There was nothing wrong about the method; we were giving value and were extinguishing debt and were not putting ourselves in a false position; we were simply sacrificing a certain amount of our property and selling to a rival country. We might be in an inferior position as a creditor nation after peace to some extent, but he did not think that involved anything of a hollow or unsatisfactory kind in relation to the currency; it did not argue a depreciated currency, for instance. It had nothing to do directly with the question of inflation. What we were suffering from was the failure to balance trade. Our exports had naturally diminished and our demand for imports was abnormally high, and that necessarily brought about a failure to balance, and we were meeting that failure by the export of securities. He saw nothing in any way wrong in that; it was the only course open to us. He thought, on the whole, Mr. Falk did not say that he was satisfied that there was inflation so far as this country was concerned, but that he simply desired an enquiry. There could be no objection to that, and it would be very desirable to know under what conditions the currency notes were issued. He had never come across anyone who could tell him what those conditions were. That was a very unsatisfactory state of things, and there was no precedent for it in our history—an issue of currency the conditions of which were not declared or understood. Even in the Napoleonic war the Banks had a very clear rule: they never issued their notes except on the discount of "good "mercantile bills, not exceeding 61 days' date, at the rate of 5 per "cent."

Passing to Mr. Kiddy's remarks, he had pointed out very well that there were plenty of explanations of the adverse exchange and of the high prices, without being driven to resort to the explanation by inflation. That there had been inflation internationally no one could doubt, enormous inflation of currency, especially by notes, and

in this country there had been a certain amount of inflation by bank credits. But the inflation was mainly an international matter. We, in this country, must hold our own; we must provide for Government in some way or other a mass of purchasing power equivalent to their needs, relative to the purchasing power enjoyed by the countries which were our rivals. He did not see that we could control or diminish in any way the purchasing power at the disposal of our Government. It was our duty to make it as large as possible. Mr. Kiddy also suggested that our currency notes, although they might not have caused inflation here, had contributed to inflation because they had been the means of enabling us to export gold to the United States, and that export of gold to the United States had been one of the principal causes of the rise of prices. That, of course, was so. On the other hand, we were driven to export gold to the United States because we had to maintain the exchange; it would be deplorable if we ceased to keep up the tradition of gold payment in this country, and he did not see how else we could have acted.

Then there was the question whether the Governments were buying well. That was a point on which he had no special knowledge, but his impression was that they had not bought as well as they might have done, and it was a matter to which attention ought to be paid. He agreed with Mr. Kiddy that at bottom it was a question for the public rather than the Government. The public must produce more and must consume less. Those were the radical facts of the position, and they were not entirely within the control of the Government, although he thought the Government should give a very distinct lead. The people were never in a more docile mood than they were to-day; they felt they were in the presence of a very difficult situation and they believed the Government was better informed than they were, and they were willing to take the word from the Government. As to continuous borrowing and the big loan, Mr. Kiddy had suggested that continuous borrowing did not appeal directly to the investor. That was so, as it was conducted in this country, but it was not as it was conducted in France. This country had not given the ordinary investor any chance; we offered Treasury bills of £1,000 minimum, which was very nice for financial people but did not appeal in any way to the ordinary investor. In France one could buy a bill for as low a value as 100 francs, or say £4. In that country they thought it politic to open all their loans in all their different forms to the very smallest subscriber, and he believed that that was a sound policy and he would like to see it done here.

Mr. Townley had spoken of Professor Nicholson's results and had quoted a sentence which he very well remembered, but which did not seem to him to be supported by any figures that Professor Nicholson had hitherto given. Mr. Townley's own enquiries seemed to have shown a curious correspondence between currency issues and subsequent prices, but he did not think Mr. Townley had mentioned any amounts but had merely referred to the direction of variation, so that that did not come to very much. It only showed a general sympathy, but unless the movements were proportional in magnitude

he did not think it could be held to show causal relation. It was suggested Government could control the currency issue, but he did not very well see how it could, though he spoke—as most people spoke—in the dark. If he thought that Government was financing the war by the issue of currency notes he should deprecate it most strongly, because he thought that that was the worst possible way of financing the war, but he did not believe that the note issue represented borrowing by Government. He did not know whether the suggestion that the notes ought to be withdrawn was seriously put forward or that it was really argued that we should return to gold circulation. If it was, we should require about another £120,000,000 of gold, and how would it be possible to finance the American exchange? The thing seemed to him absolutely impracticable. The great merit of the currency notes was that, if properly managed, we substituted for the expensive circulation of gold an inexpensive circulation of Treasury notes, and as a result were in possession of £100,000,000 to £120,000,000 of gold which we could use to settle the American exchange. The whole operation was absolutely defensible if it did not go beyond that. He was not quite clear that it might not have gone £20,000,000 beyond that, *i.e.*, beyond the amount necessary to make the substitution of paper for gold in circulation.

[In connection with the subject of this paper, reference may be made to the paper on *Statistical Aspects of Inflation*, read before the Royal Statistical Society by Prof. J. Shield Nicholson, in March last, and since published in the July number of the Journal of the Society, also to the same author's collected articles on "War Finance"—now in the press.—EDS. *J.I.A.*]

Joint-Life Annuities by two Makeham Tables with different constants; with an application to the Government Annuity Experience, 1875–1904.

THE object of this Note is to show that joint-life annuity-values based on a combination of two Makeham tables with different constants can be calculated approximately from joint-life functions for equal ages, the equated age being found by a modification of the ordinary uniform seniority method. The practical application would be to the calculation of male and female joint-life annuities when, as in the Government Annuity Experience, 1875–1904 (see *J.I.A.*, vol. xlvii, p. 66), the mortality tables for males and females respectively have been graduated by Makeham formulas with different constants.

It will be assumed that the select mortality tables are merged in ultimate tables after the first few years.

Let the constants of the two tables be k_t, s, g_t, c and K_t, S, G_t, C (the ultimate values of k_t, g_t, K_t, G_t being k, g, K, G),

let the force of mortality according to the c table be μ and according to the C table M, and let $a_{[x][y]}$, a_{xy} be the select and ultimate joint-life annuity-values on a life x subject to the mortality of the c table and a life y subject to the mortality of the C table.

Then $$a_{xy}=\Sigma v^t s^t g^{c^x(c^t-1)} S^t G^{C^y(C^t-1)}$$

and $$a_{[x][y]}=\Sigma v^t k_t/k_0 \,.\, s^t g_t^{c^{x+t}}/g_0^{c^x} \,.\, K_t/K_0 \,.\, S^t G_t^{C^{y+t}}/G_0^{C^y}$$

Let w be determined by the equation

$$\mu_x/\log_e c+M_y/\log_e C=\mu_w/\log_e c+M_w/\log_e C \quad . \quad . \quad (1)$$

so that $$C^w-C^y=(c^x-c^w)\log_e g/\log_e G$$

$$G^{C^w-C^y}=g^{c^x-c^w}$$

and $$g^{c^x}G^{C^y}=g^{c^w}G^{C^w}$$

and let $(c^w-c^x)\log_e g$, or $\mu_x/\log_e c-\mu_w/\log_e c$, be denoted by r.

Then $$a_{xy}=\Sigma v^t s^t g^{c^w(c^t-1)} S^t G^{C^w(C^t-1)} \,.\, g^{(c^x-c^w)(c^t-C^t)}$$

$$=\Sigma v^t {}_tp_{ww}\left[1+(C^t-c^t)(c^w-c^x)\log_e g + \frac{1}{2}(C^t-c^t)^2(c^w-c^x)^2(\log_e g)^2+\ldots\right]$$

(where ${}_tp_{ww}$ denotes the joint-life probability by the combined tables)

$$=a_{ww}+r\beta_{ww}+\frac{1}{2}r^2\gamma_{ww}+\ldots \quad . \quad . \quad . \quad (2)$$

where $$w \text{ is given by } (1)$$

$$r=\mu_x/\log_e c-\mu_w/\log_e c$$

$$\beta_{ww}=\Sigma v^t(C^t-c^t)\,{}_tp_{ww}$$

and $$\gamma_{ww}=\Sigma v^t(C^t-c^t)^2\,{}_tp_{ww}$$

Similarly
$$a_{[x][y]}=g_0^{c^w-c^x}G_0^{C^w-C^y}\Sigma v^t\,{}_tp_{[w][w]}\,g_t^{(c^x-c^w)c^t}G_t^{(C^y-C^w)C^t}$$

But $$g_0^{c^w-c^x}G_0^{C^w-C^y}=g_0^{r/\log_e g}G_0^{-r/\log_e G}$$

$$=1-r(\log_e G_0/\log_e G-\log_e g_0/\log_e g)$$

or $1-r(\log G_0/\log G-\log g_0/\log g)$, very nearly ;

and $g_t^{(c^x-c^w)c^t}G_t^{(C^y-C^w)C^t}=g_t^{-rc^t/\log_e g}G_t^{rC^t/\log_e G}$

$$=1+r(C^t\log G_t/\log G-c^t\log g_t/\log g)$$
$$+\frac{1}{2}r^2(C^t\log G_t/\log G-c^t\log g_t/\log g)^2+\ldots$$

Hence approximately

$$a_{[x][y]}=[1-r(\log G_0/\log G-\log g)]\,[a_{[w][w]}+r\beta_{[w][w]}+\frac{1}{2}r^2\gamma_{[w][w]}+.] \quad (3)$$

where w and r have the same values as for the ultimate annuity

$$\beta_{[w][w]}=\Sigma v^t(C^t\log G_t/\log G-c^t\log g_t/\log g)\,{}_tp_{[w][w]}$$

and
$$\gamma_{[w][w]}=\Sigma v^t(C^t\log G_t/\log G-c^t\log g_t/\log g)^2\,{}_tp_{[w][w]}$$

The values of β, γ, &c., increase somewhat rapidly as the equated age w decreases, but the successive powers of r—which then becomes small—diminish rapidly. On the other hand when r is relatively large β, γ, &c., are small. On the whole it seems probable that in most cases occurring in practice (2) and (3) would give fairly close approximations without the inclusion of terms after γ, and that if the difference between C and c were small the γ term might be neglected.

Through the kind offices of Mr. H. Weatherill, F.I.A., the values according to the Government Annuity Experience, 1875–1904, of the 3 per-cent joint-life annuities for equal ages on two male lives, two female lives, and male and female lives jointly have been supplied to us, by permission of the Controller-General, for publication in the *Journal.** These values are given in the first appended table, and in the following auxiliary tables are given the necessary values for obtaining for any combination of ages between 40 and 80 (1) the 3 per-cent values of joint-life annuities on two male lives or two female lives and (2) the approximate 3 per-cent values, by the method indicated above, of joint-life annuities on a male life and a female life. The value of $\log G_0/\log G-\log g_0/\log g$ is approximately $\cdot 033$.

The new Government tables afford a rather severe test of the suggested method of approximation, since the values of

* We were indebted in the first instance to Mr. D. S. Savory, A.I.A., for the offer of a table—which he had calculated independently—of the joint-life annuities for equal ages on two female lives, but having been led by this offer to devise the approximate method of the Note we have taken the opportunity of publishing the official tables for two lives of either sex.—Eds. *J.I.A.*

log c differ by as much as ·006—the log c of the male table being ·038 (as in the British Offices Male Table) whereas that of the female table is ·044. Nevertheless it would appear that within the published range of ages the method gives results sufficiently accurate for most purposes. The following are examples:

Ultimate Values.

Ages x	Ages y	w	r	a_{ww}	$r\beta_{ww}$	$\frac{1}{2}r^2\gamma_{ww}$	a_{xy} Approximate Value	a_{xy} True Value
45	55	49·46	−·0322	11·425	−·253	+·014	11·186	11·183
45	65	56·60	−·1184	9·136	−·133	·041	8·744	8·737
45	75	65·75	−·3460	6·203	−·426	·045	5·822	5·816
55	65	59·81	−·0845	8·089	−·214	·010	7·885	7·884
65	75	70·15	−·2204	4·907	−·152	·006	4·761	4·758
55	45	52·53	+·0312	10·455	+·178	·007	10·640	10·639
65	45	61·09	+·1120	7·674	+·244	·014	7·932	7·931
75	45	70·14	+·3210	4·910	+·221	·013	5·144	5·143
65	55	62·27	+·0820	7·296	+·155	·006	7·457	7·452
75	65	72·01	+·2135	4·396	+·114	·004	4·514	4·515

Select Values.

(w and r as for Ultimate Values.)

Ages x	Ages y	$a_{[w][w]}$	$r\beta_{[w][w]}$	$\frac{1}{2}r^2\gamma_{[w][w]}$	$\left(a+r\beta+\frac{1}{2}r^2\gamma\right) \times -·033r$	$a_{[x][y]}$ Approximate Value	$a_{[x][y]}$ True Value
45	55	11·834	−·263	+·014	+·012	11·597	11·593
45	65	9·574	−·459	·043	+·036	9·194	9·187
45	75	6·705	−·474	·049	+·072	6·352	6·345
55	65	8·547	−·230	·011	+·023	8·351	8·350
65	75	5·443	−·178	·007	+·038	5·310	5·308
55	45	10·873	+·187	·007	−·011	11·056	11·054
65	45	8·141	+·263	·014	−·031	8·387	8·387
75	45	5·446	+·260	·014	−·061	5·659	5·658
65	55	7·771	+·168	·006	−·021	7·924	7·920
75	65	4·945	+·137	·004	−·036	5·050	5·052

The irregularity of the errors is due in part to w being approximate—a source of error common to all methods based on an equated age—and probably in part to the fact that β and γ, having been calculated mainly by four-figure logarithms, are only approximately correct.

Government Annuity Experience, 1875–1904. *Value of a Joint-Life Annuity of* 1 *on Two Lives of the same age. Interest* 3 *per-cent.*

Ages		Two Males		Two Females		Male and Female	
x	x	At date of Purchase $a_{[x][x]}$	After 5 years from Purchase a_{xx}	At date of Purchase $a_{[x][x]}$	After 5 years from Purchase a_{xx}	At date of Purchase $a_{[x][x]}$	After 5 years from Purchase a_{xx}
40	40	13·871	13·563	15·351	14·853	14·559	14·162
41	41	13·591	13·281	15·100	14·605	14·293	13·895
42	42	13·307	12·994	14·843	14·349	14·020	13·622
43	43	13·020	12·703	14·578	14·087	13·742	13·344
44	44	12·728	12·407	14·307	13·818	13·459	13·061
45	45	12·432	12·108	14·030	13·543	13·171	12·771
46	46	12·133	11·805	13·746	13·261	12·879	12·477
47	47	11·831	11·499	13·457	12·973	12·582	12·179
48	48	11·526	11·189	13·162	12·679	12·280	11·876
49	49	11·219	10·877	12·861	12·378	11·976	11·568
50	50	10·911	10·563	12·555	12·073	11·667	11·257
51	51	10·599	10·246	12·244	11·761	11.355	10·943
52	52	10·285	9·928	11·929	11·445	11·041	10·625
53	53	9·975	9·609	11·609	11·125	10·725	10·304
54	54	9·662	9·289	11·286	10·800	10·407	9·982
55	55	9·350	8·969	10·960	10·471	10·087	9·657
56	56	9·038	8·650	10·631	10·140	9·766	9·332
57	57	8·726	8·331	10·299	9·805	9·446	9·006
58	58	8·418	8·014	9·966	9·468	9·125	8·679
59	59	8·111	7·699	9·632	9·130	8·805	8·353
60	60	7·806	7·386	9·297	8·791	8·486	8·028
61	61	7·505	7·077	8·962	8·450	8·170	7·703
62	62	7·207	6·770	8·628	8·111	7·855	7·382
63	63	6·913	6·468	8·296	7·773	7·543	7·063
64	64	6·624	6·170	7·965	7·435	7·235	6·747
65	65	6·339	5·876	7·637	7·101	6·930	6·434
66	66	6·059	5·588	7·313	6·769	6·630	6·126
67	67	5·785	5·306	6·992	6·441	6·335	5·823
68	68	5·517	5·030	6·676	6·117	6·045	5·526
69	69	5·255	4·761	6·365	5·798	5·761	5·234
70	70	5·000	4·498	6·059	5·485	5·484	4·949
71	71	4·751	4·243	5·761	5·179	5·212	4·670
72	72	4·510	3·996	5·469	4·879	4·948	4·399
73	73	4·275	3·755	5·184	4·586	4·691	4·135
74	74	4·048	3·523	4·906	4·302	4·441	3·880
75	75	3·828	3·299	4·637	4·026	4·199	3·632
76	76	3·616	3·084	4·376	3·759	3·965	3·394
77	77	3·412	2·877	4·123	3·502	3·739	3·164
78	78	3·214	2·678	3·879	3·254	3·520	2·943
79	79	3·025	2·488	3·644	3·015	3·311	2·732
80	80	2·842	2·307	3·418	2·787	3·109	2·529

AUXILIARY TABLES *for obtaining* (1) *the Values of Joint-Life Annuities on two Male Lives or two Female Lives,* (2) *the approximate Values of Joint-life Annuities on a Male Life and a Female Life—Government Annuity Experience*, 1875–1904. *Interest* 3 *per-cent.*

Two Male Lives:

$$a_{[x][x+h]} = a_{[x+t][x+t]}\ ;\ a_{x\cdot x+h} = a_{x+t\cdot x+t}$$

Two Female Lives:

$$a_{[y][y+h]} = a_{[y+T][y+T]}\ ;\ a_{y\cdot y+h} = a_{y+T\cdot y+T}$$

Male Life and Female Life:

$$a_{[x][y]} = (1 - \cdot 033r)\left[a_{[w][w]} + r\beta_{[w][w]} + \frac{1}{2}r^2\gamma_{[w][w]}\right] \textit{ approximately}$$

$$a_{xy} = a_{ww} + r\beta_{ww} + \frac{1}{2}r^2\gamma_{ww} \textit{ approximately}$$

where $\mu_x/\log_e c + M_y/\log_e C = \mu_w/\log_e c + M_w/\log_e C$

and $r = \mu_x/\log_e c - \mu_w/\log_e c$

Difference of Ages h	Two Males t	Two Females T	Age x	Male and Female $\mu_x/\log_e c$	$M_x/\log_e C$	$\mu_x/\log_e c + M_x/\log_e C$
1	·51	·51	40	·1240	·0968	·2208
2	1·04	1·05	41	·1280	·0986	·2266
3	1·60	1·61	42	·1323	·1005	·2328
4	2·17	2·20	43	·1370	·1027	·2397
5	2·77	2·81	44	·1422	·1050	·2472
6	3·39	3·45	45	·1478	·1077	·2555
7	4·03	4·11	46	·1540	·1106	·2646
8	4·69	4·79	47	·1607	·1138	·2745
9	5·36	5·49	48	·1680	·1173	·2853
10	6·06	6·22	49	·1760	·1213	·2973
11	6·78	6·96	50	·1847	·1256	·3103
12	7·51	7·72	51	·1942	·1305	·3247
13	8·26	8·50	52	·2046	·1358	·3404
14	9·02	9·30	53	·2159	·1417	·3576
15	9·80	10·11	54	·2283	·1482	·3765
16	10·60	10·94	55	·2418	·1555	·3973
17	11·41	11·78	56	·2566	·1635	·4201
18	12·23	12·64	57	·2726	·1723	·4449
19	13·06	13·50	58	·2902	·1821	·4723
20	13·91	14·38	59	·3094	·1930	·5024
21	14·77	15·27	60	·3303	·2050	·5353
22	15·64	16·17	61	·3531	·2183	·5714
23	16·51	17·08	62	·3780	·2330	·6110
24	17·40	17·99	63	·4052	·2492	·6544
25	18·29	18·91	64	·4349	·2672	·7021
26	19·20	19·84	65	·4673	·2871	·7544
27	20·11	20·78	66	·5026	·3092	·8118
28	21·03	21·72	67	·5412	·3336	·8748
29	21·95	22·67	68	·5834	·3606	·9440
30	22·88	23·62	69	·6293	·3904	1·0197
31	23·81	24·58	70	·6795	·4235	1·1030
32	24·75	25·54	71	·7343	·4601	1·1944
33	25·70	26·50	72	·7940	·5006	1·2946
34	26·65	27·47	73	·8593	·5454	1·4047
35	27·60	28·44	74	·9305	·5949	1·5254
36	28·56	29·41	75	1·0082	·6498	1·6580
37	29·52	30·39	76	1·0930	·7105	1·8035
38	30·48	31·37	77	1·1856	·7777	1·9633
39	31·45	32·35	78	1·2866	·8520	2·1386
40	32·42	33·33	79	1·3969	·9343	2·3312

w	Male and Female				w
	Select Values		Ultimate Values		
	$\beta_{[w\ [w]}$	$\gamma_{[w][w]}$	β_{ww}	γ_{ww}	
40	20·17	184·8	19·55	179·3	40
41	18·41	151·8	17·83	147·1	41
42	16·79	124·5	16·25	120·6	42
43	15·29	102·0	14·79	98·8	43
44	13·92	83·4	13·45	80·8	44
45	12·66	68·2	12·22	66·0	45
46	11·50	55·7	11·09	53·9	46
47	10·43	45·6	10·05	44·0	47
48	9·46	37·3	9·10	35·9	48
49	8·56	30·4	8·22	29·2	49
50	7·74	24·7	7·42	23·7	50
51	6·99	20·1	6·69	19·3	51
52	6·31	16·3	6·03	15·6	52
53	5·68	13·2	5·42	12·6	53
54	5·12	10·7	4·87	10·2	54
55	4·61	8·62	4·37	8·21	55
56	4·14	6·94	3·91	6·62	56
57	3·71	5·60	3·49	5·34	57
58	3·32	4·50	3·12	4·29	58
59	2·97	3·62	2·78	3·44	59
60	2·66	2·92	2·47	2·76	60
61	2·37	2·34	2·20	2·20	61
62	2·11	1·87	1·95	1·75	62
63	1·88	1·50	1·73	1·39	63
64	1·68	1·20	1·53	1·11	64
65	1·49	·96	1·35	·88	65
66	1·33	·77	1·19	·70	66
67	1·18	·60	1·04	·55	67
68	1·04	·48	·915	·44	68
69	·928	·37	·801	·34	69
70	·822	·29	·701	·27	70
71	·728	·23	·612	·22	71
72	·643	·18	·533	·17	72
73	·568	·14	·463	·13	73
74	·502	·11	·401	·10	74
75	·444	·09	·347	·08	75
76	·393	·07	·301	·06	76
77	·347	·06	·259	·05	77
78	·307	·05	·223	·04	78
79	·273	·04	·192	·03	79
80	·241	·03	·164	·02	80

LEGAL NOTES.

By William Charles Sharman, F.I.A., *Barrister-at-Law.*

Policy issued under Assurance Companies Act, 1909, to provide funeral expenses held to be contract of indemnity.

It is a well known legal principle that a policy of life assurance is not in general a contract of indemnity, but an important exception to this rule has recently been made in the case of *Goldstein* v. *Salvation Army Assurance Society* (1917) 2 K.B. 291.

The action was brought to recover £14 16s., under a policy of insurance entered into between the plaintiff and the defendants on 12 April 1912, upon the life of the plaintiff's mother, who died on 19 May 1916. The policy recited that a proposal had been made by the plaintiff to "effect an insurance on the life of "the assured, Sarah Goldstein, for the purpose of providing "money to be paid for the funeral expenses of the assured."

The plaintiff had maintained his mother for several years and had insured her life in various insurance societies. He alleged he had spent £53 16*s*. on her funeral expenses, which sum included £16 8*s*. 9*d*. for a tombstone. He had received from the other societies £39, and claimed £14 16*s*.—the difference between £53 16*s*. and £39.

The county court judge gave judgment for the plaintiff and the defendants appealed.

The policy was issued under s. 36 ss. (1) of the Assurance Companies Act, 1909, which includes among the purposes for which collecting societies and industrial assurance companies may issue policies, money to be paid for the funeral expenses of a parent.

It was held on appeal before Rowlatt, J., and McCardie, J., that if the appellants desired there should be a new trial before the county court judge to decide how much actually had been expended, and whether such expenses were reasonable.

In the course of the judgment Rowlatt, J. said: "The "effect of s. 36 ss. (1) of the Assurance Companies Act is not to "create a new insurable interest, but enables a person to assure "that particular disbursement which he apprehends as a moral "obligation he may have to make, and that it falls to the tribunal "before whom the claim is preferred when death happens to "enquire what that amount is. What occurs if the funeral "expenses are insured several times over it is not easy to "determine. For the present purpose it is enough to say that

"it has been decided in *Wolenberg* v. *Royal Co-operative Collecting Society*" (*J.I.A.*, vol. xlix. p. 359) "that if at the time of "action the plaintiff has already recovered under other policies "the full amount of the funeral expenses paid by him, then no "further amount can be recovered under another policy. In "the present case the plaintiff must prove that £53 16*s.* has "been spent in funeral expenses which are proper and "reasonable."

McCardie, J., said: "I agree with my learned brother that "s. 36 must be interpreted so as to effect the object of the "Legislature, which was to enable poor people by means of "insurance to provide decent funeral arrangements which "otherwise they could not themselves supply. I need not "attempt an enumeration of funeral expenses, but I think it "was quite open to the learned judge to find that the cost of an "ordinary stone or tablet upon the grave containing the body "of the dead is a funeral expense within s. 36. In my view the "learned judge that tries a case of this sort should not unduly "extend the operation of the section.

"Whether in the present case a tombstone is a reasonable "item of expense is a question of fact for the learned judge to "determine, and when he deals with the new trial he ought to "remember that the policy is a mere indemnity policy. I agree "that the sum of over £50, which was put forward by the "plaintiff as the expenses appropriate to the burial of his mother, "is a sum so large as to justify the learned judge in making a full "and critical enquiry into the facts and items and the payments "in the other cases. I agree that there should be a new trial "for the reasons given by my learned brother."

Mortgage of life interest. Trustees entitled to continue to pay income to beneficiary.

The case of *Higgins* v. *Pawson* 33 T.L.R. 233 dealt with the question whether the trustees under a settlement of personal property who have received notice of a mortgage of the life interest are entitled to continue to pay the income to the beneficiary until notice has been given them to pay the income to the mortgagee. It was decided they were entitled to do so. The facts are as follows:

One of the defendants, J. J. Pawson, was tenant for life, under a settlement of which the other defendants were the trustees, of certain funds of personal estate. In September 1914 Mr. Pawson

assigned his life interest to the plaintiff to secure a sum of money and interest thereon, the interest being payable half-yearly. On 6 January 1916, the plaintiff commenced the present action against Mr. Pawson and the trustees, claiming not only to have the security enforced by foreclosure, but to have the trustees declared personally liable for the income paid by them to Mr. Pawson from the date of the mortgage. The plaintiff in November 1914, had given the trustees the usual notice of the mortgage, and they had duly acknowledged receipt of it, but he had never given them notice or requested them to pay the income to himself.

Sargant, J., after giving the plaintiff the usual judgment for foreclosure, said that the claim against the trustees was very unusual, bold and almost preposterous. It was contended on behalf of the plaintiff that the notice to the trustees of the charge was equivalent to an entry into possession of the mortgaged life interest. Certainly none of the parties thought the notice was given with that intent. In *Dearle* v. *Hall* (1828) (3 Russ. 1, 2 L.J. (O.S.). Ch. 62) Lord Lyndhurst in his judgment described the act of giving the trustees notice as "in a certain degree" taking possession of the fund, and no authority had been produced which laid down the proposition put forward on behalf of the plaintiff. In the case of real estate, whether the estate were one in fee simple or only for life, undoubtedly the mortgagor remained in possession of the rents and profits, and his Lordship was not prepared to say, without very definite authority, that the mere giving a notice of a mortgage on the income of personal estate, without more, had any further result than in the case of a notice of a mortgage on real estate. His Lordship was of opinion that the plaintiff was not entitled to charge the trustees with payment of the income of the funds in the manner claimed; but even if he were wrong in the view he took, it would be a case in which the trustees ought to be absolved from such liability by virtue of the powers conferred upon the court by s. 3, ss. (1), of the Judicial Trustees Act, 1896.

Validity of policy on life of alien enemy.

A case of some interest, dealing with the validity of a life assurance policy on the life of an alien enemy, is that of *Seligman* v. *Eagle Insurance Co.* (1917) 1 Ch. 519. The facts are as follows:

Prior to the outbreak of war an alien effected two policies

on his own life with an insurance company and assigned them to the company by way of mortgage to secure a loan, and he and his sureties jointly and severally covenanted with the company to repay the loan with interest, and also to pay the annual premiums on the policies. On the outbreak of war, the assured became an alien enemy and left the country. Afterwards one of the sureties paid to the company the premiums as they accrued due from time to time, which the company accepted, with the reservation that they did not warrant the validity of the policies. Subsequently the surety tendered the amount due on the mortgage and claimed an assignment of the policies which the company refused to execute without a similar reservation.

It was held that while the right of the policyholder is suspended during the war, the policies were not voided merely by the assured becoming an alien enemy, that the payment to and receipt by the company of the premiums from the surety did not involve illegal intercourse with the enemy, and that the surety on payment of what was due under the mortgage was entitled to an assignment of the policies without reservation.

In the course of his judgment Neville, J., said: "The right of "the policyholder is clearly suspended during the war, and were he "to die to-morrow his executors could recover nothing from the "company, but whenever peace is restored between the countries "normal relations in this regard will be resumed, and, although the "right of the policyholder is undoubtedly suspended, if the policy "itself is not made void either at the time when war was "declared, or at the time when the current year of the policy "ran out, I can see nothing illegal in the acceptance of the "premiums by the company, because no benefit can accrue "to the enemy alien at all as the result of the payment of his "premium; but what will result is that, perhaps, some day "somebody who is not an enemy alien may have a right to sue "the company for the amount assured. It seems to me this is "one of those cases where the right is suspended.

"I come, therefore, to the conclusion that the company "were bound to hand over the securities without reservation to "the surety upon payment of the debt, and that the limitations "they propose to insert in the assignment are not justified."

ACTUARIAL NOTES.

Summation Formulæ introducing Second Difference Errors, and Formulæ correct to Third Differences derived therefrom. By A. D. WATSON, A.I.A.

IN the last number of the *Journal* (vol. l, p. 259), it was pointed out as a matter of theoretical interest that a graduation of colog p_x free from introduced error can be obtained by combining in proper proportions graduations thereof by two formulæ, each introducing a second difference error of $+\frac{1}{12}b_0$ provided the net introduced error in βc^x is positive in one graduation and negative in the other. Karup's 19-term formula and Mr. Spencer's original 21-term formula, each modified so as to introduce an error of $+\frac{1}{12}b_0$, were suggested as formulæ suitable for graduating colog p_x (without combination) on account of high smoothing power and smallness of introduced error. The results of a graduation of the O[M] New Data by the latter formula and also by the same formula with reference to colog p_x, O[M(5)], were given in comparison with a graduation of the same data by Mr. Spencer's New 21-term formula.*

It is the object of the present Note to show (1) that the introduced error of $+\frac{1}{12}b_0$ may be wholly eliminated from formulæ of this class, giving formulæ correct to third differences with substantially the same smoothing power: (2) that by eliminating a part thereof or increasing the same, formulæ free from error when applied to graduate colog p_x may be obtained; and (3) that by an extension of the same principles additional formulæ correct to third differences may be deduced. In Table IV, appended, three series of formulæ are shown, formulæ (*c*) and (*d*) of each series being constructed as suggested in (1) and (2) above, respectively. Mr. Spencer's new 21-term formula is included as it was specially designed for the graduation of colog p_x. In Table V are given formulæ constructed as suggested in (3) above.

* A balanced error occurred in the graduation and consequently the figures in Tables I and II of the Note do not compare so favourably with Mr. Spencer's as they should. It is thought unnecessary to give the corrected figures in extenso, as the characteristics of the graduations remain unchanged. A summary of the totals is, however, given in Table II of this Note, along with similar figures for formula 3 (*d*) of Table IV herein. Formula 3 (*b*) of this Note is No. 8 of the former Note.

As the substitution of $[m+1]\,[n-1]$ or $[m-1]\,[n+1]$ for $[m]\,[n]$, m and n being odd, in the operator of a summation formula, correct to third differences, introduces an error of $+\frac{1}{12}b_0$ it is evident that this error can be eliminated by adding to the operand $-\frac{1}{12}b_0$ or an equivalent, as for example, $-\frac{1}{36}(b_0+b_{\pm 1})$, which may be conveniently used in the case of the Friendly Society formula.

For illustration the Friendly Society formula, 1 (b), which is the same as Higham's formula $\frac{[5]^3}{125}\{u_0-(b_0+b_{\pm 1})\}$ with the operator modified to $\frac{[4]\,[5]\,[6]}{120}$ may be taken. To correct for the introduced second difference error the formula may be written

$$u'_0=\frac{[4]\,[5]\,[6]}{120}\left\{u_0-\frac{37}{36}(b_0+b_{\pm 1})\right\}$$

$$=\frac{[4]\,[5]\,[6]}{120}\left\{u_0-(b_0+b_{\pm 1})\right\}-\frac{[4]\,[5]\,[6]}{120\times 36}(b_0+b_{\pm 1})$$

whence it is seen that for graduation the necessary correction is easily made, and that for an investigation of the change in smoothing power, so far as this may be done without actual trial, it is only necessary to investigate the change due to $\frac{[4]\,[5]\,[6]}{120\times 36}(b_0+b_{\pm 1})$. The operand of this expression is very simple when written in terms of the coefficients of u, and consequently the investigation is not difficult. To ascertain the change in the smoothing coefficient it is, of course, necessary to correct the coefficients of $\Delta^3 u$ of the original formula by the amount of the coefficients of

$$-\Delta^3\frac{[4]\,[5]\,[6]}{36}\{b_0+b_{\pm 1}\}$$

before squaring.

The same observations hold, *mutatis mutandis*, for formulæ 2 (b) and (c), and 3 (b) and (c).

It will be noticed on comparison of formulæ (b) and (c) in each of the three series in the Table IV appended that the increase in the negative error, when $u_x=\text{colog}\,p_x$, due to the elimination of all the positive error, is but a small percentage of the positive error so eliminated. Since for any given value

of $\log c$ the positive (second difference) error is proportional to the coefficient of $\frac{d^2}{dx^2}$, it is a simple matter to calculate the percentage of the full second difference error which must be retained in order that positive and negative errors may balance when $\text{colog}\, p_x$ is graduated, assuming that equal $\frac{\text{decreases}}{\text{increases}}$ in the positive error are accompanied by equal $\frac{\text{increases}}{\text{decreases}}$ in the total negative error. The formula may be expressed as follows, namely, {Total error factor formula (c) less net error factor formula (b)}x=Total error factor, formula (c).

As an illustration we have from formulæ 1 (b) and 1 (c), taking $\log c = {\cdot}039$, $(448920 + 242297)x = 448920$, whence $x = 64{\cdot}95$ per-cent. In a similar manner all the percentages shown below were calculated.

TABLE I.

Showing the percentage of the full second difference error to be retained in order to obtain a formula in which positive and negative errors balance when $\text{colog}\, p_x$ is graduated.

No. of Formula in Table IV	Value of $\log_{10} c$		
	·038	·039	·040
1 (b)	61·70	64·95	68·28
2 (b)	74·65	78·58	82·60
3 (b)	118·53	124·74	131·10

In the case of formula 1 (d) 64 per-cent was taken, for by so doing the operand assumes the simple form $u_0 - {\cdot}01(b_0 + b_{\pm 1})$ which is easy to apply. In the case of formula 3 (d) 124·72 per-cent was used so as to obtain a terminal decimal in the correction factor, simplifying slightly the investigation of smoothing power. It will, however, be observed that the unbalanced error, formulæ (d)—$\log c = {\cdot}039$—can, to the last place of decimals, be accounted for by the fact that the percentages obtained by the above formula were not used. Therefore, for any given value of $\log c$ it is an easy matter to modify a formula involving a positive second difference error so as to eliminate all error in $\text{colog}\, p_x$; and when any other function is to be graduated the full second difference error

can be eliminated, resulting in a formula involving an error of the fourth and higher orders. In the case of an actual graduation there would usually be no object in departing from the theoretical correction factor, for any assumed value of $\log c$, unless to take advantage of a simpler form as in the case of the Friendly Society formula. Generally, the most suitable value of $\log c$ will not be known, and if at a first trial the difference between the actual and expected deaths is material a change in the correction factor can be made, and if the graduation is an important one a third trial should make the actual and expected deaths agree almost exactly. It is not suggested that this should be done in practice or that more satisfactory results would thus be arrived at than in some such way as suggested by Mr. Spencer (*J.I.A.*, vol. xli, p. 380).

As would be expected there is no great difference in smoothing power between formulæ (*b*), (*c*) and (*d*) of each of the three series. Formula 3 (*d*) shows slightly increased smoothing power over 3 (*c*), due to the fact that it is necessary to further increase the second difference error in order to balance the negative error due to the fourth and higher differences. For the graduation of colog p_x a formula constructed after the manner of 3 (*d*) would appear to be specially suitable. A summary comparison of graduations of colog p_x O^M New Data by three formulæ shown in Table IV is given below.

Table II.

Item compared	Formula No. in Table IV			
	4	3 (*b*)	3 (*b*) with reference to $O^{M(5)}$ as standard	3 (*d*)
Deviation—expected deaths less actual ...	+217	+237	+244	+242
	−222	−243	−242	−242
Accumulated deviation, quinquennial groups ...	+82	+102	+116	+106
	−199	−195	−143	−157
Sum of $10^5\Delta^3 q_x$ ...	+183	+175	+173	+175
	−126	−120	−118	−120

From the above it will be noted that by formula 3 (*d*) the actual and expected deaths agree exactly. Formula 3 (*b*) with reference to $O^{M(5)}$ as standard would probably, in general, give

quite as good results as 3 (*d*), and is perhaps slightly easier to apply.*

To obtain formulae (*d*) Table IV the correction factors used were determined on the assumption that $\log_{10}c = \cdot 039$, but the introduced errors, positive and negative, are shown for the same formulae when $\log c = \cdot 038$ and $\cdot 04$, to enable an opinion to be formed as to the importance of the introduced error in $\operatorname{colog} p_x$ when the true average value of $\log c$ throughout may differ from the value assumed.

In most tables which have been Makehamized, the value of $\log c$ used falls within the limits ·038 and ·040. Nevertheless the true values applicable to various sections of the same table would in all probability show much wider variations. In a cognate connection Mr. Spencer remarks (*J.I.A.*, vol. xli, p. 405) that "the absolute magnitude of the error would probably not "be increased in the same proportion, as the increase in c "would in all likelihood be accompanied by a decrease in "β or B." To add interest to this point there is given in the following table a comparison of the values of q_x deduced from the constants calculated by Mr. King (*J.I.A.*, vol. xli, p. 532) from the O^{M5} unadjusted data, ages 80 to 98, with the values of q_x after making allowance for the error which would be introduced if Mr. King's values of $\operatorname{colog} p_x$ were graduated by formulae (*d*) and 4 of Table IV. The q's of the official graduation are also given.

TABLE III.

Age	Official Graduation	King's Constants	Formula in Table IV			
			1 (*d*)	2 (*d*)	3 (*d*)	4
80	·13850	·15577	·15572	·15570	·15568	·15583
85	·20569	·20091	·20084	·20082	·20077	·20103
90	·30075	·27861	·27849	·27846	·27837	·27882
94	·40000	·37431	·37413	·37409	·37396	·37460

It will be noted that the introduced error is in all cases trifling in comparison with the departure of the official

* The correction to be made to $\operatorname{colog} p_x$ when graduated by formula No. 3 (*b*) of this Note with reference to $\operatorname{colog} p_x$, O^{M5}, are to be found in *J.I.A.*, vol. l, p. 265, where the formula is designated as formula No. 8. The values of βc^x are also given so that the corrections for any other formula in Table IV may similarly be obtained by multiplying βc^x by the error factor shown in said table for $\log_{10}c = \cdot 039$. The values of βc^x at ages 20 and 74 should be ·0002812 and ·0362744, respectively.

graduation from that by Mr. King's constants, notwithstanding that Mr. King's value of $\log c$ is ·059605 as against ·039 assumed in the construction of formulæ (d).

In the same manner in which formulæ (c) were derived from formulæ (b) by modifying the operand of the former, so also formulæ correct to third differences may be obtained from any operator combined with any operand by modifying the said operand so as to eliminate the error which would otherwise remain in the formula. General summation formulæ have long been available, but it has been recognized that only a very limited number of particular formulæ based thereon were practically useful. Perhaps nearly all, except the well-known formulæ, would lack facility of application and the essential quality of smoothing power. It would be extraordinary if this were not so, bearing in mind the limitations that the operator and operand must be so chosen that when the formula is expressed in terms of u_0 and its central differences the positive and negative coefficients of b_0 will exactly balance, the coefficients in the operand being integral and convenient in application. When, however, the operand is modified as already illustrated many formulæ may be obtained reasonably facile in application with the arithmometer and giving excellent results in comparison with formulæ of equal range heretofore published. A few of these formulæ are given in Table V. To avoid confusion they are numbered consecutively with those in Table IV. It is not suggested that these are the best formulæ which can be obtained in this way: they are given as examples only.

No. 5 may be compared with Mr. Spencer's 15-term formula.

No. 6 should give substantially better results than No. 2 (a) (Dr. Karup's 19-term formula) being almost identical with No. 2 (c). The sum of the squared coefficients in the operand of No. 6 is 1·45 against 1·4 in No. 2 (a). The operand might also be written

$$\left\{2u_0+\frac{37}{48}(u_{\pm 1}-u_{\pm 3})\right\}$$

the sum of the squared coefficients being 1·59, but the smoothing power would not be quite so good.

No. 7 compares favourably with 3 (c) and is rather easier to apply.

No. 8 has high smoothing power for a 21-term formula,

(Negative Quantities in Italics.)

	*3 (*a*)	$\frac{\lfloor 5 \lfloor 2 \lfloor 7}{350}$	$2u_0-(2b_0+2b_{+1}+b_{+2})$ or $2u_0-(u_{+1}+u_{+3})$	2		…	…	…	·766	·114	$\frac{1}{160}$
				1	*12·6̇0*	*738529*	*819392*	*906720*			
				6	*34·857144*	*15642*	*18280*	*21279*			
				8	*48·39250̇*	*166*	*205*	*251*			
				Total error factor		*754337*	*837877*	*928250*			

Rule	n	$\dagger \phi n$	$\log_{10} c = 0.98$	$\log_{10} c = 0.99$	$\log_{10} c = 0.10$?	Sum of the coefficients of the five central terms	Sum of the third differences of the coefficients	Smoothing coefficient
$u_0 - (\delta_0 + \delta_1 \ldots)$ or $\ldots(-u_{-1} - u_1 \ldots)$	2					872	320	$\frac{1}{56}$
	4	6.10	875126	116199	[illegible]			
	6	12.6	[illegible]	6641	[illegible]			
	8	12.160	13	54	[illegible]			
	Total error factor		[illegible]	[illegible]	[illegible]			
Ditto	2	[illegible]	637195?	672016	706920	850	167	$\frac{1}{95}$
	4	[illegible]	81069?	122793	467849?			
	6	13.1011.5740?	5879?	6871	7998			
	8	12.9122?	44	55	67			
	Negative error factor		186912?	129749?	47591?			
	Nett		[illegible]	242267?	241004?			
$u_0 - \frac{37}{36}$ (Ditto)	2					855	169	$\frac{1}{93}$
	4	6.29...	[illegible]	141760	188841			
	6	13.51630	[illegible]	7104	[illegible]			
	8	1?.3500	16	56	69			
	Total error factor		[illegible]	148920	197149?			
Ditto	2	[illegible]	408317	130090?	[illegible]	852	168	$\frac{1}{94}$
	4	[illegible]	[illegible]	129621	175408?			
	6	[illegible]	[illegible]	[illegible]	[illegible]			
	8	[illegible]	15	65	65			
	Negative error factor		[illegible]	130031?	183572?			
	Nett		[illegible]	6441?	51143?			
$(3h_0 + 4h_1 + 2h_2)$ or $(5u_0 - 2u_1 \ldots 3u \ldots + 2u_2)$	2					843	167	$\frac{1}{106}$
	4	[illegible]	[illegible]	[illegible]	[illegible]			
	6	17.51?	7873?	9201?	10740			
	8	19.867..?	68?	84?	103?			
	Total error factor		467126?	516?28?	572116?			
Ditto	2	·083	637996?	672016	706920	827	147	$\frac{1}{120}$
	4	[illegible]	46?128?	[illegible]	568529?			
	6	18.0416?	8400?	9466?	11019			
	8	[illegible]	[illegible]	[illegible]	105?			
	Total error factor		171299?	[illegible]	[illegible]			
	Nett		166636?	148627	127195?			
Ditto	2					831	149	$\frac{1}{118}$
	4	[illegible]	[illegible]	[illegible]	792486?			
	6	[illegible]	[illegible]	[illegible]	1157?			
	8	[illegible]	[illegible]	[illegible]	110			
	Total error factor		[illegible]	[illegible]	[illegible]			
$5u - \ldots$ (Ditto)	2	[illegible]	401337?	528070?	[illegible]	828	147	$\frac{1}{120}$
	4	[illegible]	467278?	[illegible]	[illegible]			
	6	[illegible]	8136?	[illegible]	11006?			
	8	[illegible]	[illegible]	88?	107			
	Negative error factor		[illegible]	[illegible]	[illegible]			
	Nett		[illegible]	[illegible]	[illegible]			
[illegible]						786?	111?	$\frac{1}{160}$
		[illegible]	[illegible]	[illegible]	[illegible]			
		[illegible]	[illegible]	[illegible]	[illegible]			
		[illegible]	166?	29?	241?			
		[illegible]	[illegible]	[illegible]	926250?			
Ditto		[illegible]	[illegible]	672016?	706920?	756?	107	$\frac{1}{172}$
		[illegible]	[illegible]	[illegible]	914016?			
		[illegible]	[illegible]	[illegible]	[illegible]			
		[illegible]	[illegible]	[illegible]	[illegible]			
		[illegible]	[illegible]	[illegible]	[illegible]			
$2u - \ldots$ Ditto		[illegible]	[illegible]	[illegible]	913501?	759?	109	$\frac{1}{170}$
		[illegible]	[illegible]	[illegible]	22100?			
		[illegible]	[illegible]	[illegible]	[illegible]			
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	107	$\frac{1}{179}$
		[illegible]	[illegible]	[illegible]	[illegible]			
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	777	167	$\frac{1}{141}$

but the operand is a little complicated. It may, however, be written

$$\left\{\frac{68}{24}u_0 + u_{\pm 1} + u_{\pm 2} - \frac{23}{12}u_{\pm 3}\right\}$$

which would not take much additional time to apply.

$$\left\{3u_0 + \frac{51}{52}(u_{\pm 1} + u_{\pm 2} - 2u_{\pm 3})\right\}$$

might also have been chosen as operand giving a smoothing coefficient of $\frac{1}{172}$, the sum of third differences ·110 and the sum of the 5 central coefficients ·764.

Nos. 9 to 13 are 23-term formulæ showing varying degrees of wave-cutting and pressure on the humps and hollows, passing into formulæ of high smoothing power. These formulæ suggest the extent to which the principles herein outlined may be applied to produce a formula suitable for any particular purpose.

Nos. 14 and 15 are given as examples of 25-term formulæ of high smoothing power. In 15 it will be noted the coefficient of the central term in the operand is zero. This in some measure flattens the coefficient curve.

Nos. 16 and 17 are given as examples of 27-term formulæ with high smoothing power, being substantially higher than Mr. Kenchington's and with less error. However, they bring less pressure on the waves.

To apply a formula of the type suggested the necessary modification of the operand should not usually take more than one hour, and in some cases the advantage may be considered worth the additional time. It is interesting to note the great variety of formulæ which can be derived from modifications of the operand of Mr. Spencer's original 21-term formula, namely $(2u_0 + u_{\pm 1} - u_{\pm 3})$, which when sufficiently modified as herein indicated becomes $2(u_0 + u_{\pm 1} - u_{\pm 3})$ and passes through all the other variations shown. In fact, with this one operand formulæ correct to third differences of 19 terms and over may be produced, giving rather more satisfactory results than the usual summation formulæ hitherto published, although generally slightly more laborious to apply.

Table V.

Classification and Analysis of formulæ.

o.	Formula	Number of Terms	Coefficient of $-\frac{d^4}{dx^4}u_0$	Sum of coefficients of five central terms	Sum of third differences irrespective of sign	Smoothing coefficient
5	$\frac{[3][4][6]}{72}\left\{u_0+\frac{29}{36}(u_{\pm1}-u_{\pm2})\right\}$	15	4·49	·901	·279	$\frac{1}{64}$
6	$\frac{[4][5][6]}{240}\left\{u_{\pm1}+\frac{43}{54}(2u_0-u_{\pm3})\right\}$	19	8·15	·832	·146	$\frac{1}{118}$
7	$\frac{[4][6][7]}{168}\left\{u_0+\frac{49}{96}(u_{\pm1}-u_{\pm3})\right\}$	21	13·11	·759	·097	$\frac{1}{170}$
8	$\frac{[4][6][7]}{504}\left\{-u_0+u_{\pm1}+u_{\pm2}+\frac{23}{24}(4u_0-2u_{\pm3})\right\}$	21	13·55	·756	·108	$\frac{1}{176}$
9	$\frac{[3][4][12]}{144}\left\{u_0+\frac{83}{96}(u_{\pm1}-u_{\pm3})\right\}$	23	37·00	·465	·247	$\frac{1}{80}$
0	$\frac{[3][5][11]}{165}\left\{u_0+\frac{76}{96}(u_{\pm1}-u_{\pm3})\right\}$	23	30·66	·541	·164	$\frac{1}{111}$
1	$\frac{[4][5][10]}{200}\left\{u_0+\frac{69}{96}(u_{\pm1}-u_{\pm3})\right\}$	23	25·10	·605	·135	$\frac{1}{146}$
2	$\frac{[5][6][8]}{240}\left\{u_0+\frac{61}{60}(u_{\pm2}-u_{\pm3})\right\}$	23	20·52	·694	·118	$\frac{1}{160}$
3	$\frac{[5][6][8]}{240}\left\{u_0+\frac{61}{96}(u_{\pm1}-u_{\pm3})\right\}$	23	19·25	·701	·076	$\frac{1}{249}$
4	$\frac{[6][7][8]}{336}\left\{u_0+\frac{73}{96}(u_{\pm1}-u_{\pm3})\right\}$	25	26·28	·664	·060	$\frac{1}{267}$
5	$\frac{[6][7][8]}{672}\left\{u_{\pm2}+2\frac{1}{48}(u_{\pm1}-u_{\pm3})\right\}$	25	27·28	·660	·066	$\frac{1}{292}$
6	$\frac{[6][7][10]}{420}\left\{u_0+\frac{91}{96}(u_{\pm1}-u_{\pm3})\right\}$	27	39·83	·592	·055	$\frac{1}{367}$
7	$\frac{[6][8][9]}{432}\left\{u_0+\frac{89}{96}(u_{\pm1}-u_{\pm3})\right\}$	27	37·83	·613	·044	$\frac{1}{385}$

Surrender Values of Altered-Class Policies. By ALEXANDER FRASER, F.I.A., F.F.A.

CASES are now-a-days frequent where the class of assurance changes during the currency of a policy, such as alterations from whole-life to endowment assurances, or from pure endowment to whole-life or endowment assurance, as in the children's deferred assurances now so common. The quotation of surrender-values of such policies after alteration is often irksome, unless a regular scale has been worked out, as care must be taken to quote for some years a value both fair in itself, and consistent with the surrender-value before conversion, and this may involve the comparison of several different calculations. The following general method may be found useful in many cases, as it is simple and may be modified so as to fit in with the various rules of offices in their quotations.

The surrender-value immediately after conversion should be approximately equal to the value before conversion, and if the plan is adopted of making it *exactly* the same, the effect is to give an equation from which may be found the net premium for use during the remainder of the policy's existence. The equation is $V' = A - \pi'\mathbf{a}$, whence $\pi' = (A - V') \div \mathbf{a}$, where V' is the surrender-value before conversion, and A and $\mathbf{a}$ are the respective assurance and annuity-values in the altered policy. For instance, neglecting bonuses, if a policy is changed at age x from a whole-life to an endowment assurance for m additional years

$$\pi' = \frac{A_{x\overline{m}|} - V'}{\mathbf{a}_{x\overline{m}|}} = \pi_{x\overline{m}|} - \frac{V'}{\mathbf{a}_{x\overline{m}|}}$$

The value of the policy after n further premiums have been paid will then be

$$A_{x+n\,\overline{m-n}|} - \left(\pi_{x\overline{m}|} - \frac{V'}{\mathbf{a}_{x\overline{m}|}}\right)\mathbf{a}_{x+n\,\overline{m-n}|}$$

$$= {}_nV_{x\overline{m}|} + V'\frac{\mathbf{a}_{x+n\,\overline{m-n}|}}{\mathbf{a}_{x\overline{m}|}}$$

$$= {}_nV_{x\overline{m}|} + V'(1 - {}_nV_{x\overline{m}|})^*$$

* The Editors point out that similar formulas have been given by Mr. A. K. Blackadar (*Transactions* Actuarial Society of America, Vol. iv, p. 26), for use in connection with returns under the Canadian Life Insurance Act, 1877, but suggest that, notwithstanding, the insertion of this Note in the *Journal* may be convenient for reference.—A.F.

Therefore the policy may be valued by the ordinary rules of the office as a new assurance effected at age x, with the addition of the value before conversion multiplied by the complement of the new policy value. The expression may usually be more conveniently written

$$V' + (1 - V')_n V_{\overline{xm}|}$$

Where a reversionary bonus has been added before conversion, its original and increased values are added to the two sides of the original equation for V', so that the increase in the value of the bonus at date of conversion will form a constant deduction from V' in the later formulas.

The same method applies in the case of children's deferred assurances with guaranteed options, but as in this popular though somewhat unsatisfactory scheme competition seems to have forced the cash options at age 21 up to rather high figures, a small percentage deduction from these options for each premium after that age for 5 or 10 years may be desirable, so that the values may be gradually brought into line with those under the ordinary classes. As bearing on this point it may be mentioned that the use of the unmodified cash option in some cases gives a negative value for π' when the rate of interest used in calculating surrender-values is $4\frac{1}{2}$ per-cent, which seems to indicate that these options are in reality too large when compared with the usual practice of offices.

It is not intended that, in valuing the assurance as a new policy from date of conversion, the common allowance for initial expenses should again be made by omitting the first year of assurance or otherwise; this has already been provided for in the original calculation of V'.

The principles above described may be applied with equal facility to the calculation of paid-up policies, and of the altered net premiums to be used for valuation purposes.

Care must of course be taken that the principle is applicable to the particular class of alteration with which one is dealing. For instance, if an assurance has been altered from whole-life to whole-life by limited payments, the formula for the value of the altered policy is ${}_{n:m}V_x + V'(1 - {}_nV_{\overline{xm}|})$; the multiplier of V' must obviously vanish after m years, as the value of the policy is then simply A_{x+m}.

REVIEWS.

The Combination of Observations. *By* DAVID BRUNT, M.A., B.Sc.

(Pp. x + 219. Price 8s. Cambridge University Press.)

THE arrangement of this book follows that of mathematical text-books in giving the more important propositions, proofs and explanations, and by means of a few examples the kind of problem that can be solved is indicated. Some of the examples are worked out completely, while others are left to the reader.

After an introductory chapter the author gives Hagen's proof of the "error law" and a generalized form of that proof due to Professor Eddington, and he then discusses the form of the curve and the calculation of values for it.

From this he proceeds naturally to the evaluation of measures of precision, mean square errors and probable errors, making the well-deserved remark with regard to the last-mentioned term that it is an unhappy one because the most probable error is zero corresponding to the highest point of the error curve. His examples on this part of the work are interesting, but they suffer to some extent from relating to small series of observations, so that the warning given in respect of example 1, pp. 39, &c., really applies to many of the others. His discussion of systematic errors is interesting.

The next step in the work is to deal with observations of different weights and the warning that he gives with regard to arbitrary scales of weighting, that there is a general tendency to regard bad observations as worse than they really are, is one that it is important to bear in mind in practical work; in fact, generally speaking, our feeling about weighting is that it is very difficult to choose an entirely satisfactory method: in most work that we have ourselves come across we have found it best to leave arbitrary weightings severely alone. Some good examples are given at the close of the chapter dealing with this part of the subject and also some of a more general character.

The author next deals with the adjustment of observations involving more than one unknown and explains carefully the formation of the normal equations, &c., and then discusses their solution. The methods of substitution (Gauss and Doolittle) are given in full, with an example, and the method of determinants is indicated, but the author remarks in a footnote that this method is not very practical. We think, however, that it might be made so by forming from the table of the determinant a table in the same form giving the logarithms of the quantities in the determinant. By this means we should replace a number of multiplications by additions, and with a little practice we think this could be made almost automatical. We are bound to admit, however, that the solution of the normal equations in any form involves a large amount of work, and must necessarily be tedious when there are several unknowns.

Following this part of the work the author gives a short chapter in which he discusses the adjustment of conditioned observations.

It may, perhaps, be worth while to indicate a case in which a "condition" arises in actuarial work. If we wish to make a graduation of a select table of mortality run into a graduated ultimate table after n years by a method such as that used by Sir George Hardy for the $O^{[NM]}$ Table, we must choose our constants for the period before selection has worn off, so that the adjustment to the ultimate rates of mortality becomes zero at the end of n years. This is a definite condition which must be fulfilled by the constants.

The part of the work dealing with the more ordinary propositions of the normal curve of error and the adjustment of observations by least squares closes with a short chapter on the rejection of observations, and while the author indicates certain criteria that might be of help, his general conclusion may be fairly summarized by the word "don't"; with this general conclusion we are in entire agreement, although everyone who has dealt with statistics must on rare occasions have had to discard some observation because he is convinced that it is not in reality an "observation."

Having thus dealt with what we might call the normal curve aspect of the problem, Mr. Brunt turns to alternatives to the normal curve and refers particularly to the Pearson curves and mentions also those of Thiele, Edgeworth and Charlier. He gives no examples, but his short treatment seems to us sufficient for the purpose which he had in view, and it is certainly satisfactory to find that an author who approaches the subject from what we may perhaps call the more academic side, appreciates the limitations of the normal error curve and where its application will break down in practical work. In the early chapters this is indicated in more than one place, and while it is not a point that is likely to be overlooked by actuaries, it is certainly one which some authors have not sufficiently accentuated in the past.

The next chapter deals with correlation and contingency, and seems to us to explain the methods of calculation and the theory in a clear and interesting manner.

The two final chapters of the book deal with harmonic analysis from the point of view of the method of least squares with the object of investigating periodicity. If we are endeavouring to represent a recurring rise and fall by a curve, we naturally turn to trigonometric functions, such as $\sin\theta$ or $\cos\theta$ because they give a simple example of recurrence when the value of θ is increased by 2π. The form which has actually been used in astronomical work is

$$l = A_0 + A_1\cos\theta + B_1\sin\theta$$

for a single periodic term where θ is given a range from 0 to 2π or

$$\begin{aligned} l = A_0 + A_1\cos\theta + A_2\cos 2\theta + \ldots \\ + B_1\sin\theta + B_2\sin 2\theta + \ldots \end{aligned}$$

where there is more than one period.

The author shows how the method of least squares is used to find the A's and B's, and how the work can be simplified in certain special

cases, and he also discusses the unmasking of hidden periodicities by means of the periodogram. His examples will greatly help the reader to follow the methods.

Periodicity may arise in statistics which come before the actuary, but the subject, although fascinating, has received little attention from members of our profession. We hope that Mr. Brunt's interesting chapters will stimulate actuarial students not only to try the methods he gives, but also to endeavour to find alternatives.

The book is a valuable attempt to set out the methods used for the combination and adjustment of observations, and ought to become popular with those who are interested in the subject. It is well arranged, very readable, and an excellent example of the almost perfect printing of the Cambridge University Press.

We may add that it is a sign of the times that the preface is written from General Headquarters.

W. P. E.

On a Formula facilitating the application of Select Mortality for all durations. By J. F. STEFFENSEN, Ph.D.

(Svenska Aktuarieföreningens Tidskrift, 1917.)

THIS is a sequel to a paper published some ten years ago ("Notes on the practical graduation of life insurance tables". Transactions Fifth International Actuarial Congress, vol. ii, p. 247), in which Dr. Steffensen showed that $1/\bar{a}$—or $1/\mathring{e}$—by an aggregate table could be represented, with sufficient accuracy for practical purposes, by the formula $\alpha + \beta c^x$ (log c being normally ·026 and, like the log c of Makeham's formula for μ, varying little from table to table), and that, since log $_tp_x$ can then be expressed explicitly in terms of α, β, c and δ, the constants could be determined—and graduated continuous annuity-values (or expectations) calculated—direct from an ungraduated experience. Whether this would save as much arithmetical work as Dr. Steffensen anticipated, and would consequently facilitate the application of a new ungraduated experience to life assurance finance, may be open to question. Given graduated values of colog $_xp$ (which could apparently be obtained as readily, with the same degree of accuracy, as those of $1/\bar{a}$) the calculation of a complete annuity table by the formula $\log a_x = \log vp_x + \log a_{x+1}$ is quite a small matter. Moreover, as there is no simple relation between annuity-values at different rates of interest, it would be necessary, in order to obtain graduated annuity-values at other rates of interest, (1) to make an independent graduation for each rate, or to determine the constants for each rate in such a way as to secure the best possible general agreement of the μ's (μ being expressible explicitly in terms of $\bar{a}$, α, c and δ), in either of which cases the annuity values at different rates would correspond to as many slightly different mortality tables, or (2) to construct a mortality table from the graduated annuity values at one rate, and then to calculate annuity values at other rates in the usual way, in which

case any saving of work effected by the calculation of the annuity values direct from the experience would be lost. Apart, however, from the applicability of the method to the calculation of tables for practical purposes, its by-products are extremely interesting. Dr. Steffensen showed that if $1/\bar{a} = \alpha + \beta c^x$, many other functions in addition to the two already mentioned can be simply expressed in terms of $\bar{a}$ and the constants, *e.g.*, the temporary annuity-value and the policy-value, also that $\bar{a}$ can be expressed in terms of μ by solution of a quadratic; and since it has been shown that this formula for $1/\bar{a}$ does in fact represent its value approximately in an aggregate table, it would appear that an auxiliary table of $\bar{a}$ might be very useful in analytical work and in obtaining convenient approximations to the values of complicated functions.

In the present paper Dr. Steffensen discusses the application of the method to select tables. The object of this second paper is essentially practical, for having come to the conclusion that in life assurance work account will eventually have to be taken of selection throughout life, Dr. Steffensen thinks that actuaries will be compelled to simplify their technical apparatus and so avoid the labour of constructing monetary tables (as in the case of the Swedish experience) for every age and duration. This involves the necessity for a simple formula—lending itself to the explicit expression of such functions as V and $a_{x\overline{n}|}$—for $a_{[x]+t}$, and it was, perhaps, with a view to its contingent advantages in this respect that Dr. Steffensen was led to adopt experimentally the formula $1/\bar{a}_{x+t} = \alpha + \beta c^{x+t} - \kappa c^x - \lambda c^t$, which gives for log ${}_tp_{[x]}$, $\mu_{[x]+t}$, V and $\bar{a}_{x\overline{n}|}$ expressions very similar to, although naturally rather more complicated than, those obtained in the case of an aggregate table. Dr. Steffensen tests the formula by graduations of the O[M] ungraduated expectations (Principles and Methods, p. 146) and the new Swedish $3\frac{1}{2}$ per-cent. annuity-values (*J.I.A.*, vol. l, p. 59). In these graduations no attempt has been made to secure the best possible fit, but they are quite sufficient to illustrate the capabilities of the formula. The results show, as Dr. Steffensen states, a "very fair general agreement" between the graduated and ungraduated values, but we hardly think they justify the view that the formula "is able to render in broad outlines the most important features of selection." It is certainly a curious and interesting fact that the graduation of the O[M] expectations should reproduce the feature noticed by Sir George Hardy in the ungraduated experience, namely, that in certain cases $e_{[x]+t}$ is > $e_{[x+n]+t-n}$. But it is much more material that the formula should reflect more or less correctly the effect of selection in the early years, and this, as it seems to us, it entirely fails to do. In discussing the minima and maxima of the formula for $\mu_{[x]+t}$—that is to say, of the formula derived from $1/\bar{a}_{[x]+t} = \alpha + \beta c^{x+t} - \kappa c^x - \lambda c^t$—Dr. Steffensen finds that in the graduation of the Swedish experience a minimum of μ for x constant, arises only when x is $< 27{\cdot}4$ and t (in the case of $x = 27$) $= {\cdot}4$, and he infers that the minimum is of no practical importance. We doubt whether the inference is correct. The minimum itself may be unimportant, but the fact that it exists at 27 and that

there is presumably an approach to it at older ages, seems to indicate that the slope of μ is all wrong for the early values of t. This would appear to be the case from inspection of the observed and graduated values of $\mu_{[x]+t}$ in the Swedish experience, the rise in μ in the first five years being on the average less than one-fourth as steep in the latter as in the former, and it is shown also by the following comparison of the experimental and the O$^{[M]}$ values of $\mu_{[x]+t}$:

x	From Experimental Graduation of $1/\bar{a}$		Hardy's Graduation	
	$t=0$	$t=5$	$t=0$	$t=5$
30	·01085	·00987	·00121	·00707
50	·00909	·01555	·00503	·01711
70	·05268	·08679	·02805	·07766

Too much stress must not, of course, be laid on the incidence of a maximum or minimum at a particular point in an experimental graduation, but the comparative flatness of μ at the start would appear to be a feature of the suggested method, and if that is the case it is obvious that a graduation by this method would be inapplicable to any short term purposes, and would result in an under-estimate of the select policy-value in the early years. The formula for μ has also a minimum for t a constant and not large. In the aggregate graduation of the *Text-book* $3\frac{1}{2}$ per-cent Table, given as an example in the earlier paper, the minimum was at $x=26{\cdot}85$, and this might be passed as unobjectionable in view of other experience although there is no justification for it in the *Text-book* table. But the select formula seems to place the minimum at altogether too late an age—at any rate in the tables experimentally graduated by Dr. Steffensen. In the case of the Scandinavian table the minimum of $\mu_{[x]}$ is found at 42·8, and in the O$^{[M]}$ graduation it appears to be at 43·4. It is only right to say that Dr. Steffensen anticipates that the simple four constant formula might not be found sufficiently accurate for a definite graduation, and that better agreement, without disturbance of the convenient properties of the formula, might be secured by adding one or two more terms.

In the concluding part of the paper Dr. Steffensen shows that if $1/\bar{a}_{[x]+t}$ can be represented with sufficient accuracy by the formula given above, an approximate select valuation could be made by using a "limiting" table defined by $1/\bar{a}_{x+t}=\alpha+\beta c^{x+t}$ and adding a correction of the form $f(x)\ \bar{a}_{x+t}-\phi(x)\ \bar{a}^2_{x+t}$. The $f(x)$ and $\phi(x)$ could be scheduled with the other particulars in writing up the classification, and the correction (which would apparently take a somewhat simpler form if the select net premium were valued by the "limiting" table) could then be obtained for each attained age without sub-grouping according to duration. A similar method, except that the correction would have to be limited to policies of

less than 10 years' duration, might, perhaps, be employed to obtain an approximate $O^{[M]}$ valuation, for from the functional similarity which Dr. Steffensen has found to exist between μ and $1/\bar{a}$ it would seem probable that for the principal ages at entry ${}_tV_{[x]}$ $(t<10)$ might be represented approximately by some such expression as ${}_tV'_x - a^2_{x+t}\,[(10-t)+m(10-t)^2]\phi(x)$ which could be written in the form ${}_tV'_x - a^2_{x+t}\,[\phi_0(x)+(x+t)\,\phi_1(x)+(x+t)^2\phi_2(x)]$, where V' denotes the value by the ultimate table with select net premiums.

Although Dr. Steffensen's investigations do not appear to us to have reached, at present, an entirely satisfactory conclusion, the two papers will repay careful study. They are, incidentally, models of research work, combining ingenuity of conception and dexterity in analysis with lucidity of exposition.

Su una relazione fra l'annualità vitalizia di gruppo e l'annualità semplice, nell'ipotesi di Makeham. *By* Prof. F. Insolera.

(Atti della Reale Accad. delle Scienze di Torino. Vol. lii. 1916–7.)

This tract recalls an almost forgotten episode in the history of actuarial research—the publication of Makeham's Table of the Integral of Gompertz's Function (*J.I.A.*, vol. xvii, pp. 312–319), and subsequently of Emory McClintock's rival method of integrating the function by expanding it in powers of $c^x \log_e 1/g$ (*J.I.A.*, vol. xviii, p. 242). There can be little question now as to the relative merits of the two methods. McClintock's expansion is of some interest academically, but its practical applicability is limited even in the case of one life (*J.I.A.*, vol. xxxiii, p. 312), and still more when two or more lives are involved; moreover, if applicable, it involves a laborious calculation. Makeham's Table, on the other hand, was perhaps somewhat too ambitious in scope, for it applies to any aggregate Gompertz (or Makeham) function, whatever its constants, and consequently entails an amount of calculation which would be unnecessary if it had been constructed for a particular set of constants: it has no heading, and is unaccompanied by any simple rules for its use in practice; it may have been, to some extent, discredited by an unimportant mistake—subsequently corrected by Makeham (*J.I.A.*, vol. xvii, p. 445)—as to the range required to make it theoretically complete: at any rate it has seldom, if ever, been used, and it suffered the final indignity of being incorrectly described in the index to the First Forty Volumes. Nevertheless, it was a bold and modern method of dealing with the problem, a method worthy in every way of its author; it introduced into actuarial science the principle, generally recognized by mathematicians, that an intractable definite integral, to be of practical use, must be tabulated, so that any desired value can be found by interpolation. If the property of uniform seniority had not been available, a table of the integral of the Makeham function (supposing the Makeham formula to have been employed in that event for graduating standard

mortality tables) would have been as necessary to actuaries as tables of the gamma-function and the probability-integral are to mathematicians and statisticians: and even now a table on similar lines to Makeham's, adapted to a standard mortality table such as the *Text-book* or the $O^{M(5)}$ and constructed for values of n which would admit of the integrals of functions containing the factors $(rs^m)^t$ and $(rs^m c)^t$ for $m=2$ and 3 being obtained without a double interpolation might be very useful for the evaluation of contingent assurance functions.

In the tract under notice Prof. Insolera has used McClintock's method (although without apparently having been aware of McClintock's connection with it) for the purpose of obtaining the ratio of $a_{xyz}\ldots$ to a_x, but the resulting expression, even in its most general form, appears to be subject to precisely the same limitation as McClintock's, namely, that it is impracticable when Σc^x is large, owing to the slow convergency of the expansion. Prof. Insolera deduces a simpler expression which it is suggested would render unnecessary the tabulation of joint-life annuity-values for equal ages; but this expression—involving a rough approximation to the value of c^t—gives a fair result only for young lives and seems to be inapplicable (the numerator becoming negative) to such a practical case as the evaluation of $a_{60.60.60}$ by the *Text-book* $3\frac{1}{2}$ per-cent Table. It is possible that a practicable relation might be established by McClintock's method between $a_{xyz}\ldots$ and a_w where $c^w=\Sigma c^x$, for in that case the numerators of the terms in the two expansions become identical. But such a relation seems hardly worth establishing. Joint-life annuity-values for equal ages can be tabulated with very little trouble, since it is merely necessary to substitute $n \log p$ for $\log p$ in the ordinary continued process, and if they are not available there remains the alternative method of substituting a single life and raising the rate of interest (*Text-book*, Part II, Second Edition, p. 212). A simple first difference interpolation by this method gives 15·412 for the value of $a_{30.30}$ by the *Text-book* $3\frac{1}{2}$ per-cent Table (the true value being 15·399), whereas Prof. Insolera's approximate formula gives, as the result of a somewhat laborious calculation, the rather less accurate value 15·372.

THE INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—Session 1916-1917.

First Ordinary Meeting, 27 November 1916.

The President (Mr. S. G. Warner) in the Chair.

The President delivered an Inaugural Address.

Second Ordinary Meeting, 29 January 1917.

The President (Mr. S. G. Warner) in the Chair.

Mr. Walter Borland, F.F.A., was elected an Associate of the Institute, and Dr. Johan F. Steffensen was elected a Corresponding Member of the Institute.

Mr. Hartley Withers delivered an Address entitled "Problems of Taxation."

The following gentlemen took part in the discussion:—Sir Gerald Ryan, Mr. Geoffrey Marks, and the President.

Third Ordinary Meeting, 26 March 1917.

The President (Mr. S. G. Warner) in the Chair.

Prof. H. S. Foxwell, M.A., delivered an Address entitled "Inflation: in what sense it exists; how far it can be controlled."

The following gentlemen took part in the discussion:—Messrs. O. T. Falk, A. W. Kiddy (a visitor), E. W. Townley, and the President.

The Seventieth Annual General Meeting, 4 June 1917.

The President (Mr. S. G. Warner) in the Chair.

The proceedings at the Annual General Meeting will be found on page 327.

REPORT, 1916–1917.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1916–1917, the sixty-ninth year of its existence.

There has been a *decrease* of 21 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-one years later, at 31 March 1906, it was 922. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1907	248	303	383	22	956
1908	253	313	421	22	1,009
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1917.

	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1916 .	308	345	247	17	917
ii. Withdrawals by					
(1) Death . . .	5	3	7	...	25
(2) Resignation or otherwise . .	...	3	7	...	
	303	339	233	17	892
iii. Additions to Membership					
(1) By Election . .	...	...	...	1	4
(2) By Examination .	...	...	...	...	
(3) By Re-instatement	...	3	...	...	
	303	342	233	18	896
iv. Transfers					
(1) By Examination:					
from Associates	...	...	...	...	...
to Fellows .	...	...	...	...	...
	303	342	233	18	896
(2) By Examination:					
from Students	...	...	2	...	...
to Associates .	...	2	...	...	...
v. Number of Members in each class on 31 March 1917 .	303	344	231	18	896

There are also 173 candidates admitted as Probationers, and 67 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1911 have been as follows:

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1912	181	59	1915	188	72
1913	197	55	1916	172	73
1914	200	67	1917	173	67

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of six Fellows, Messrs. T. G. Ackland, R. G. Gregson Ellis, F. W. Frankland, E. McClintock, G. M. Reeve, and J. Stocks; seven Associates, Messrs. S. O. Benjamin, G. E. Burrows, A. J. Cook, A. J. C. Fyfe, L. F. Hawkins, J. H. Marlin, and H. Wallis; and nine Students, Messrs. W. S. Emery, F. J. Grant, G. H. Grantham, E. C. Kemp, R. J. Ledger, M. E. Lobb, J. V. McLean, S. F. Snowdon, and E. R. Williamson. Seventeen of these Members, namely, Naval Instructor

[*Continued on page* 326.]

Dr. *Revenue Account for the*

1916. £	s.	d.		£	s.	d.	1917. £	s.	d.
10,386	11	7	Amount of Funds at the beginning of the year— General Fund (including Stock of Publications, other than *Journal*)	8,190	4	8			
409	0	0	Messenger Legacy Fund	421	5	5			
351	8	1	Brown Prize Fund	361	18	11			
767	15	9	G. F. Hardy Memorial Fund	767	15	9			
11,911	18	5					9,741	4	9
			Subscriptions—						
920	17	0	Fellows	792	15	0			
676	4	0	Associates	590	2	0			
247	16	0	Students	157	10	0			
93	19	6	Probationers	36	4	6			
1,938	16	6		1,576	11	6			
...			Fines on Reinstatement	1	1	0			
1,938	16	6		1,577	12	6			
618	9	0	*Less* Waived and returned to Members and Probationers on Naval and Military Service	105	10	6			
1,320	7	6					1,472	2	(
			Entrance Fees—						
5	15	6	Students	2	2	0			
13	2	6	Probationers	2	2	0			
18	18	0					4	4	(
116	19	2	Balance of Publications Account				125	16	:
			Dividends and Interest—						
319	19	9	General Fund	245	1	1			
12	5	5	Messenger Legacy Fund	12	12	9			
10	10	10	Brown Prize Fund	10	17	2			
...			G. F. Hardy Memorial Fund	25	19	6			
342	16	0					294	10	(
£1[illegible],713	19	1					£11,637	17	(

Publications Account for th

£	s.	d.		£	s.	d
202	2	5	Stock (excluding *Journal*) at the beginning of the year	297	15	[illegible]
114	10	0	*Cost of Revised Edition of Text-Book, Part I*	...		
14	4	3	Binding and Advertising	12	2	[illegible]
116	19	2	Balance	125	16	:
£477	15	10		£435	14	:

Balance Sheet

£	s.	d.	LIABILITIES.	£	s.	d.	£	s.	d.	£	s.	d
8,190	4	8	General Fund				8,045	3	7			
233	9	2	Messenger Legacy Fund	233	9	2						
187	16	3	Accumulated Dividends	200	9	0						
421	5	5					433	18	2			
200	0	0	Brown Prize Fund	200	0	0						
161	18	11	Accumulated Dividends	172	16	1						
361	18	11					372	16	1			
767	15	9	G. F. Hardy Memorial Fund	767	15	9						
.			Accumulated Dividends	25	19	6						
767	15	9					793	15	3			
										9,645	13	[illegible]
12	16	8	Sundry unpaid Accounts							17	6	:
£[illegible],754	1	5								£9,662	19	:

year ending 31 *March* 1917. **Cr.**

1916. £ s. d.		£ s. d.	1917. £ s. d.
	Journal—		
369 3 9	Printing of Nos. 265, 266	220 9 5	
71 5 0	Editorial Expenses	48 8 0	
440 8 9		268 17 5	
168 12 6	*Less* Sales during the year	112 16 9	
271 16 3			156 0 8
31 6 4	Library—Binding, Purchases, &c.		19 8 6
36 5 0	Meetings		41 14 9
...	Legal Charges		18 11 3
293 14 10	*Examination charges*	...	
135 9 0	*Less Fees received from Candidates (1915)*	...	
158 5 10			...
241 10 0	*Tutors for classes in Parts* I *and* II *for 1914-15 and 1915-16*		...
126 0 0			
600 0 0	Office Expenditure—Rent	600 0 0	
419 2 6	Salaries	488 17 2	
63 19 11	House expenses	66 19 3	
29 0 9	Fire and other Insurance	39 14 5	
89 11 8	Stationery and Printing	69 4 9	
33 2 7	Postage and Telegrams	23 17 9	
5 8 3	Sundries	11 7 10	
1,270 5 8			1,300 1 2
130 0 0	*Cost of Bust of Sir George Hardy, K.C.B.*		...
1,833 5 3	*Amount written off in respect of decrease in value of Stock Exchange Securities*		...
...	Loss on Sale of Stock Exchange Securities		456 8 1
9,741 4 9	Amount of Funds at the end of the year as per Balance Sheet		9,645 13 1
£13,713 19 1			£11,637 17 6

Examined and found correct, 1 *May* 1917.
ROBT. S. B. SAVERY,
GEORGE H. LAWTON,
W. MOUAT JONES, } *Auditors.*

year ending 31 *March* 1917.

£ s. d.		£ s. d.
180 0 3	Sales (excluding *Journal*)	186 14 0
297 15 7	Stock (excluding *Journal*) at the end of the year	249 0 2
£477 15 10		£435 14 2

Examined and found correct, 1 *May* 1917.
ROBT. S. B. SAVERY,
GEORGE H. LAWTON,
W. MOUAT JONES, } *Auditors.*

31 *March* 1917.

£ s. d.	ASSETS.	£ s. d.
1,875 0 0	£3,000 Natal 3 per-cent Inscribed Stock	1,875 0 0
726 5 0	£1,000 Dominion of Canada $3\frac{1}{2}$ per-cent Registered 1930–50 Stock	726 5 0
700 0 0	£1,000 New South Wales $3\frac{1}{2}$ per-cent Inscribed 1930–50 Stock	700 0 0
357 15 0	£600 Belgian Government 3 per-cent Sterling Loan of 1914	357 15 0
...	£4,500 5 per-cent War Stock, 1929–47	4,275 0 0
297 15 7	Stock of Publications (excluding *Journal*) in hand	249 0 2
...	Cash on Deposit Account	200 0 0
527 12 4	Cash on Current Account and in hand	441 16 4
61 19 0	Subscriptions in Arrear	70 7 0
767 15 9	£769. 15*s*. 6*d*. $4\frac{1}{2}$ per-cent War Stock, 1925–45, "B" G. F. Hardy Fund	767 15 9
874 10 0	*£1,200 Metropolitan Railway $3\frac{1}{2}$ per-cent Debenture Stock*	...
1,662 10 0	*£2,000 Great Eastern Railway 4 per-cent Debenture Stock*	...
686 5 0	*£1,000 Great Northern Railway Preferred Ordinary Stock*	...
1,216 13 9	*£1,350 Great Western Railway $4\frac{1}{2}$ per-cent Debenture Stock*	...
£9,754 1 5		£9,662 19 3

Examined and found correct, 1 *May* 1917.
ROBT. S. B. SAVERY,
GEORGE H. LAWTON,
W. MOUAT JONES, } *Auditors.*

H. Wallis, Captains R. G. Gregson Ellis and G. M. Reeve, Lieutenants A. J. C. Fyfe, E. C. Kemp, R. J. Ledger, J. H. Marlin, J. V. McLean, S. F. Snowdon, and E. R. Williamson, M.C., Lance-Corporal M. E. Lobb, Bombardier S. O. Benjamin, and Privates G. E. Burrows, W. S. Emery, F. J. Grant, G. H. Grantham, and L. F. Hawkins, have fallen in the service of their King and Country; as also have four Probationers of the Institute, Captain L. Davies, Lieutenants H. D. S. Fromant and D. Kerr, and Private P. J. Davis. To the relatives of all who have thus made the supreme sacrifice for the nation, the Council have sent letters of sympathy, feeling sure that in doing so they have represented the feeling of all Members of the Institute.

By the death of Mr. Ackland the Institute has lost one of its most valued and energetic Fellows. He was a member of the Council at the time of his death, and had served thereon for eighteen years. Besides filling the offices of Vice-President and Treasurer, Mr. Ackland was Editor of the *Journal* from 1905 to 1911, and had rendered valuable services to the Institute as Lecturer and Examiner. It will also be remembered that he was Honorary Supervisor of the Joint Mortality Investigation undertaken by the Institute and the Faculty of Actuaries twenty years ago.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,176. 6*s*. 0*d*., as compared with £1,339. 5*s*. 6*d*. received in the previous year. The Income and Expenditure for the year were £1,896. 12*s*. 9*d*. and £1,536. 16*s*. 1*d*. respectively.

The value of the Stock Exchange Securities have been written down on three occasions during the last decade. The Council felt that the time had now come to reconsider the list as a whole. It was accordingly resolved to dispose of the Railway Stocks and to re-invest in the 5 per-cent War Loan issued in January last. The resulting changes are shown in the Balance Sheet, and the loss on the sale of the securities previously mentioned appears as a charge in the Revenue Account.

The number of Members and Probationers on the roll of service with the Army and Navy has, since last year, increased to 368. The Council have to deplore the loss of 35 who have been killed in action or died of wounds.

For the same reasons which obtained last year, it has not been possible to hold the usual monthly Sessional Meetings. The Council resolved, however, to depart from the usual practice by arranging for two special Sessional Meetings to be addressed by eminent authorities on finance and economics, who are not Members of the actuarial profession. Addresses were delivered on 29 January and 26 March, by Mr. Hartley Withers and Professor H. S. Foxwell, M.A., respectively, and the large attendance at these meetings indicated that the experiment had been amply justified. The Council hope that it may be possible to make somewhat similar arrangements next year if the national conditions do not permit of the resumption of the ordinary activities of the Institute.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
27,714	Parts of *Journal*.
736	Index to Vols. 1 to 40.
1,778	*Text-Book*, Part I (Revised Edition).
44	*Text-Book*, Part II (Second Edition).
635	Government Joint-Life Annuity Tables.
734	Select Life Tables.

No. of Copies	Description of Work
39	A Short Collection of Actuarial Tables (New Edition).
988	Frequency-Curves and Correlation (W. P. Elderton).
38 *in cloth* / 2,320 *in paper*	Lectures on Finance and Law (Clare and Wood Hill).
1,527	Lectures on the Companies Acts (A. C. Clauson).
1,194	Lectures on the Law of Mortgage (W. G. Hayter).
708	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,419	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
907	Lectures on Stock Exchange Investments (J. Burn).
1,508	Lectures on Friendly Society Finance (Sir A. W. Watson).
315	South African War Mortality (F. Schooling and E. A. Rusher).
259	Life Assurance Law (A. R. Barrand).
660	British Offices' Valuation Tables.
658	British Offices' 2¾ per-cent Temporary Annuity Values.
639	Transactions of the Second International Congress of Actuaries.
788	Index to Transactions of Seven International Actuarial Congresses.
1,500	Examination Questions, 1911-15.

8 *May* 1917.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventieth Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 4 June 1917, Mr. Samuel G. Warner (the President) in the Chair.

Mr. Joseph Burn (Honorary Secretary), read the notice convening the Meeting and the Minutes of the Sixty-ninth Annual General Meeting, held on the 5 June 1916, which were confirmed. The Report and Accounts were taken as read.

The President, in moving the adoption of the Report and Accounts, said that the only feature of outstanding interest in the accounts arose in connection with the sale of some railway stocks which the Institute had held for a good many years. The Council had invested the proceeds in the National War Loan, which, they would no doubt agree, was a proper and patriotic course.

The report began, as usual, with a statistical comparative statement which called for little comment. There was a slight falling-off in the number of members, but it was not quite so great as the falling-off in the previous year. Then came a paragraph which would be read with great regret—recording the deaths of members during the year. Seventeen of these members had fallen in the service of their King and Country; as also had four probationers of the Institute. To the relatives of all who had thus made the supreme sacrifice for

the nation, the Council had sent letters of sympathy, feeling sure that in doing so they represented the feelings of all members of the Institute. He had reason to know that these letters had been received with great appreciation. It was a long list, and a very mournful list, including members of every class—Fellows, Associates, Students and Probationers—but it was a glorious and honourable list; and they would all—having the honour and reputation of the Institute at heart—feel proud and gratified in the deepest sense that, among the other institutions of the country, their Institute had so well done its share in the great work with which the nation is confronted.

The next paragraph of the report dealt with Mr. Ackland's death, which they also much regretted, and with his great and varied services to the Institute. He had had an opportunity at the first meeting of the Session of referring to the loss which the profession had sustained by Mr. Ackland's death. He much regretted to announce that Mr. David Alexander Bumsted had died on the preceding day. Mr. Bumsted was a very old member of the Institute; he became a Fellow by examination in 1866, in which year he and Mr. Manly were the only two Fellows by examination. He had now died at the age of 77. He made several interesting contributions to their discussions, especially those on reversionary subjects, in which he was particularly interested. He retired from active practice in 1905, after 50 years' service with the General Reversionary Company. A great many of them had had the pleasure of Mr. Bumsted's acquaintance; they knew him as a good, sound actuary, and a good friend, a man interested a good deal in the social side of life also, deeply respected in the Institute, deeply respected, also, as he happened to know, in the neighbourhood in which he resided. In his actuarial career he followed rather a quiet path, concerning himself chiefly with those branches of the science which came before him in the course of his official duties. After his retirement he never lost such opportunities as presented themselves of meeting his late professional friends again in a social way, and they all appreciated that and lamented his loss.

The final paragraph in the report dealt with the chief incident of the session, the holding of two special meetings. He thought that the Council had reason to congratulate themselves on the success of those meetings. It was difficult to know what course to take to prevent the interest of the members from lapsing owing to the suspension of the ordinary meetings, and it seemed to him that they had hit upon a plan which had answered the purpose, and had also given them an opportunity of coming into contact with other minds, with men of large experience in other branches of finance, and thereby gaining a great deal of information and a great deal of pleasure. As the report suggested, the Council hoped, if the same state of things continued next winter, to be able to do something of the same kind. It was not easy, of course, and might not be practicable, to make the necessary arrangements. However, two meetings in a session was not a very great number, and as they had succeeded fairly well last year, he hoped if the necessity arose, they might succeed equally well again.

There were one or two matters of general interest that he would like to refer to. One was a subject which had been occupying their minds a good deal in connection with the military situation, and about which they felt some amount of responsibility, at all events as regards watching what was done—the subject of demobilization. Of course they did not know when that would become an immediately practical question, but every month that passed brought it nearer, and as so many students connected with this Institute had, at the call of their country, gone out to the field of action, and thereby inevitably suffered a great interruption of their studies, it was their duty to urge that at the earliest possible moment when the national needs permitted, these men who had already suffered this considerable postponement of their study—which in any case had to be conducted in hours

of leisure after office work—should be able to come back and resume their studies, both in their own interests and in the interests of the profession. They had made representations to that effect as forcibly as the circumstances would allow, and, having done so they must leave the matter in the hands of the authorities. In order to avoid overlapping, they had acted in agreement with the associations representing the insurance companies—of which most of their students were employés—and he thought there was a fair prospect of their reasonable representations receiving practical attention.

He had received an interesting communication from Mr. Peters, of the Liverpool Victoria Approved Society, with reference to the excellent scheme for supplying educational books to prisoners of war. That work had been taken up very energetically indeed by Mr. A. T. Davies of the Education Department. He (the President), had had occassion once to visit the Department, and had been very much surprised to see the number of books that had already been collected, and to hear of the number that had been sent away. He learnt also from evidence shown to him, how much this was appreciated, and how detailed and intelligent were the enquiries for books of various kinds which came from the prisoners. Mr. Peters wrote in his letter: "I have just submitted a list of educational books in my possession for the use of British prisoners of war in Germany, and am delighted to find that they are also suitable, as evidenced by their acceptance. It has thence occurred to me that many other members of the Institute may be able and willing to help the studies of our soldiers, so unfortunately placed, by doing likewise. Perhaps the Council will publish a request for lists to be submitted." He went on to say that particulars could be obtained from Mr. A. T. Davies, and that he himself would be pleased to assist, and would be willing to take charge of and forward all lists submitted, and subsequently receive the books named in the list. The scheme of the Education Department had met with a considerable measure of success and deserved every support. Matriculation examinations for the University of London had been successfully held in the internment camp, and he understood that seven men had qualified. There had been also examinations held in connection with the University of Edinburgh and other Educational bodies. He welcomed the opportunity of mentioning it at the Annual Meeting and thus of giving it additional publicity.

There was one other matter about which he would say a word in a somewhat guarded way. In the ordinary course of the proceedings at that meeting the voting list for the members of the new Council would be submitted, and that procedure would be the procedure that had been followed for many years past, by which a list of the entire number of names, no more and no less, was submitted as a balloting paper for election. That scheme had worked very well in practice. Perhaps, like some things in the constitution of our country, it had worked better in practice than, from the point of view of theory, it altogether deserved to. Therefore one might say there was no particular reason for interfering with it. It had seemed to the Council, however, that something might be done to introduce a certain element of larger choice at these annual meetings, so far as new members of the Council were concerned, that a change of that nature might add to the interest of the meeting, and might further bring it home more to the members of the Institute, that they were, in a real and practical sense, consulted in the matter. He thought it would be possible to evolve some scheme which would provide that element, although it might involve the technical process of an alteration of the bye-laws. The Council had, at all events, made up its mind on this point, that even if an alteration had to be faced, a change might be beneficial and might be introduced. It seemed to all of them that the present occasion was not an unsuitable one for such a thing to be done. If it were necessary to call one or two special meetings to deal with the subject, it would at all events help, during the suspension of the ordinary sessional meetings, to maintain their interest in

the proceedings of the Institute. As regards the fact that a great many of the younger members were absent, he thought, considering how obviously any such change would work in favour of those younger members, that particular point need hardly be considered. The change would not be a very drastic or revolutionary one, but it would give a greater freedom of choice, modelled largely on the practice of institutions of a like nature to their own.

The publication of the Birthday Honours had drawn their attention that day to the recognition of public service. He would like to mention one such recognition which they were not likely to hear of otherwise, and which would be a source of gratification to all. Sir Alfred Watson, who had hitherto held the title of Actuary to the National Health Commission, would for the future be known as the Government Actuary. He had Sir Alfred Watson's permission to make this announcement on one rather difficult condition, about which he must exercise his own discretion, as to how far he should observe it, namely, that he should make no mention of himself personally. They would all agree that it was a great step in the public recognition of the Institute, and the recognition of the value of actuarial science to the country—a great and permanent step, that such a post should be created. A thing of that sort once done was done for ever, and the very fact that the nation had an official called the Government Actuary, would be a reminder that actuarial science was a very important factor in a great deal of the legislation designed for the national welfare, and that large financial schemes involving elements of interest and mortality, if launched without competent actuarial advice, might end in disappointment and disaster. The nation by such an appointment was secured from such happenings, and therefore in the national as well as the professional interest it was a subject for congratulation that such an appointment had been initiated. With regard to that member of our body who was now going to fill it, he had been asked to say nothing at all. However, he proposed to say, without enlarging on the subject, that they were all very heartily glad to see Sir Alfred Watson in that position. There was no man they would more gladly see in it now. They knew the work he had done for the nation already. They recognized that he had been acting as Government Actuary, without the title, for some time, in connection with the great national health scheme, and they all hoped that he would long live to fulfil the duties of his office, as he was uniquely qualified to do. They congratulated him as an actuary, and above all as a friend, on what had happened.

They had come to the end of a very strenuous year, and a year which had made great demands upon them. It had made demands on most of them in the way of a great deal of extra work and anxiety because of those who were away. It had been also marked by anxiety for the nation on the great issues which were now hanging in the balance, and in the case of many of them by personal anxiety and perhaps personal grief also. Probably there was no one of them who was not interested in some very peculiar sense in some member or members of the Army or Navy on active service. They looked forward to the coming year, with the hope that in the course of it there might be removed this nightmare of trouble, anxiety and tear. At all events they looked forward to it as those who were prepared, with the rest of the country, to go forward with unflinching resolution, with unshaken courage, and with a stable faith that in the end the cause of righteousness and freedom would be triumphantly established.

Mr. L. F. Hovil, in seconding the resolution, referred to the scheme for supplying educational books to prisoners of war, and said he had had the pleasure of attending, on behalf of the President, a conference on the subject during the session. The Committee which had charge of the matter evidently did extremely good work for the prisoners, and at the conference he agreed, on behalf of the Institute, to the name of the Institute being included,

in a circular to the prisoners of war, in a list of educational bodies, which, without in any way lowering the standard of examinations, would, as far as possible, allow for any work they had done under supervision in the camps. There had been extensive and complete arrangements for proper lectures under qualified persons in those camps, and it was felt that such work should be taken into account, so that students might feel that part of their time was not entirely wasted while they had the misfortune to be prisoners.

The motion for the adoption of the report and accounts was carried unanimously.

Election of Officers.

A ballot was then taken for the election of the President, Vice-Presidents, Council and Officers for the ensuing year; and the Scrutineers, Mr. W. A. Sim and Mr. C. F. Peters, subsequently reported that the following Fellows recommended by the Council had been elected:

President.

Samuel George Warner.

Vice-Presidents.

Robert Ruthven Tilt.
Ralph Todhunter, M.A.
Arthur Digby Besant, B.A.
Joseph Burn.

Council.

Samuel John Henry Wallis Allin.
Arthur Digby Besant, B.A.
Joseph Burn.
Frederick Timothy Mason Byers.
Charles Ronald Vawdrey Coutts.
William Palin Elderton.
Oswald Toynbee Falk, B.A.
Duncan Cumming Fraser, M.A.
Charles William Kenchington.
Owen Kentish.
*Abraham Levine, M.A.
Geoffrey Marks.
George Ernest May.
*Henry Edward Melville.
*William Peyton Phelps, M.A.
Sir Gerald Hemmington Ryan.
Richard George Salmon.
*William Charles Sharman.
John Spencer.
*Edward Robert Straker.
Alfred Charles Thorne.
Robert Ruthven Tilt.
Ralph Todhunter, M.A.
Samuel George Warner.
Sir Alfred William Watson.
James Douglas Watson.
Arthur Thomas Winter.
Ernest Woods.
William Arthur Workman.
Frank Bertrand Wyatt.

* New Members of the Council.

Treasurer.

Sir Alfred William Watson.

Honorary Secretaries.

James Douglas Watson.
Abraham Levine, M.A.

Mr. R. L. Elderton proposed the re-election of the Auditors, Messrs. G. H. Lawton and W. M. Jones, and the election of Mr. E. W. Humphry to take the place of the retiring auditor, Mr. R. S. B. Savery.

Mr. G. H. Maunder seconded the motion, which was carried unanimously.

Mr. A. B. Adlard, in proposing a vote of thanks to the President, the Vice-President, the Members of Council, the Auditors and the Assistant

Secretary, for their services during the past year, said he had a personal recollection of all the 24 presidents who had occupied the chair, and had very great confidence in saying that the position had never been more worthily filled than by Mr. Warner, whose personal friendship he had enjoyed for many years. Everyone would agree that Mr. Warner had proved himself to be in the right place as President by his qualifications, his attainments, his very happy temperament, his unvaried courtesy and his eloquence. His presidential address last November was a comprehensive survey of the progress and development of actuarial science, and the practical interests involved, and as Mr. Warner had said in the latter part of his address, those interests would undoubtedly increase in the future. He had also advised the younger members of the Institute to study friendly societies, and it was certain that when some of the younger members returned to this country they could not do better than act on the advice the President had given them. His own early experience in years gone by of the study of friendly societies had shown him how interesting it was, and how it led to the consideration of many other subjects which were of vital importance to the well-being of the community.

The thanks of the members were also due to the Vice-Presidents, the Council and the Officers. The activities of the Institute had necessarily been very much restricted, but those gentlemen had very heartily supported the President, and had performed their duties in an admirable manner. The Honorary Secretaries deserved special mention, as so much of the work, and a great amount of responsibility, always rested upon them. The work of the Assistant Secretary (Mr. Jarvis) also should be recognized. Before the war Mr. Jarvis was assisted by a clerk, but that clerk had been to the front and was now invalided home and in hospital. The result had been that Mr. Jarvis had done the clerk's work as well as his own, and had carried out his duties most efficiently, and to the complete satisfaction of the President and of the Council.

Mr. A. G. Hemming, in seconding the motion, added his personal congratulations to the President on his address, and said the members appreciated the happy selection made by the Council of the two papers that had been read at the sessional meetings.

The motion was carried with acclamation.

The President, in returning thanks on behalf of his colleagues and himself, said that so far as his own small part had gone, his tenure of office, although it had happened at a trying time, had been so far one of great pleasure. Again he had proved, as he had often done in the past, the joy and value of sympathetic and tolerant friendship, and of the support of men whom he esteemed and with whom he considered it an honour to serve. The outstanding thing in his recollection was the way in which the members had attended at the two meetings which had been held. It was an experiment, and he had been afraid of the first meeting, but when the hall was so well filled, the Council was encouraged to go on to a second gathering, when the hall was filled still better than at the first. That showed that the Council was getting support, and encouraged them to go on doing the best they could for the members.

There were but few changes in the Council, as the bulk of that body, according to the constitution, remained the same, but from those who in due rotation retired the members parted with extreme regret, while at the same time they had the opportunity of greeting some new members to the Council and some old friends who had come back again. He cordially welcomed Mr. Levine to his (the President's) old post of Secretary, and also cordially welcomed some younger members who had come upon the Council. The Council were trying in every possible way to promote friendship and mutual confidence and free communication between the older members of the Institute and the younger ones. They recognized what an able set of young men were rising up, and also that it would be on their shoulders

that the future work of the profession would fall and therefore desired to establish and consolidate and intensify friendly relations.

Mr. F. L. Collins proposed a vote of thanks to the Auditors (Mr. R. S. B. Savery, G. H. Lawton and W. Mouat Jones) for their services during the past year.

Mr. R. C. Simmonds seconded the motion, which was carried with unanimity.

Mr. G. H. Lawton briefly replied, on behalf of his brother auditors and himself, and the proceedings then terminated.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1916:

By whom presented (when not purchased).

Accountants and Auditors, Society of Incorporated.
List of Members, &c., 1916–17. — *The Society.*

Accountants, Institute of Chartered, in England and Wales.
List of Members, 1917. — *The Institute.*

Acerboni (A.).
Fundamentos Matèmaticos de los Seguros Sociales. 8vo. Buenos Aires. 1916. — *The Author.*

Actuarial Society of Sweden.
Transactions, 1916–17. — *The Society.*

American Mathematical Society.
Transactions, 1916–17. — *The Society.*

American Statistical Association.
Transactions, 1916–17. — *The Association.*

Association of Life Insurance Medical Directors of America.
Abstract of the Proceedings, 1889 to 1906, 1907 to 1912, 1913 to 1915, and 1915 to 1916. Four vols. 8vo. New York. 1906, 1912, 1915, 1917. — *The Association.*

Ball (W. W. R.).
Mathematical Recreations and Essays. 6th Edit. 8vo. 1914. — *Purchased.*

"Biometrika."
Volume XI, Part IV. — *Purchased.*
Containing *inter alia*—
"On the distribution of the correlation coefficient in small samples. (Appendix II to their Papers)", by "Student" and R. A. Fisher.
"On the representation of Statistical Data", by L. Isserlis.
"Relation of the Mode, Median, and the Mean in Frequency Curves", by A. T. Doodson.

Brunt (D.).
The Combination of Observations. 8vo. Camb. 1917. — *Purchased.*

By whom presented (when not purchased).

Casualty Actuarial and Statistical Society of America.
Proceedings, 1916–17. — *The Society.*

Chartered Insurance Institute, Journal of the
Vol. 19. 8vo. 1916. — *The Institute.*
Containing *inter alia*—
"State Insurance and Workmen's Compensation", by W. E. Gray.
"The Construction of a Mortality Table from population figures", by R. Thodey.

Denmark.
Beretning fra Forsikringsraadet for aaret 1915, 1916. — *The Danish Government.*

Economic Society (Royal).
Journal of the, 1916–17. — *Purchased.*

Ford (Dr. L. R.).
The effect of a rise in prices upon the amount of small money used. Edinburgh. 1917. — *The Author.*

France.
Loi du 21 Juillet 1909. relative aux conditions de Retraite du Personnel des Grands Réseaux de Chemin de Fer d'intérèt général. Examen des projets de Réglements de Retraites présentés en exécution de la loi. Rapport de l'ingénieur en chef. Fol. Paris. 1909.
Rapport fait au nom de la Commission chargée d'examiner la proposition de loi relative à la réglementation du travail des mécaniciens, chauffeurs et agents des trains et aux conditions de retraite du personnel des chemins de fer français. Par M. Paul Strauss. 4to. Paris. 1909. — *The French Government.*

Gephart (W. F.).
Principles of Insurance. Vol. I. Life. 8vo. New York. 1917. — *Purchased.*

Gumbel (E. J.).
Eine Darstellung statistischer Reihen durch Euler. 1916. — *The Author.*

Hammond (H. P.).
Life Insurance in Groups, 1912–1917. St. Paul, Minn. 1917. — *The Author.*

Hoffman (Dr. F. L.).
The sanitary progress and vital statistics of Hawaii. Newark, N.J. 1916. — *The Author.*

Holland.
Archief voor de Verzekerings-Wetenschap. 1916–17. — *The Society.*
Mededeelingen der Vereeniging voor Levensverzekering. — *The Society.*
Dordrecht Levensverzekering-Maatschappij. Verslag over het jaar 1916. — *The Company.*
Nationale Levensverzekering-Bank. Verslag over het jaar 1916. — *The Company.*

By whom presented (when not purchased).

Institute of Actuaries Students' Society.

"Valuation of Liabilities and Distribution of Surplus." Notes, Papers and Discussion arranged by the Society. 8vo. 1916. — *Purchased.*

Institute of Bankers.

Journal of the, 1916–17. — *The Institute.*

Insurance Institute of America.

Proceedings of the Eighth Conference. 8vo. 1916. — *The Institute.*

Insurance Institute of Toronto.

Proceedings, 1916–17. — *The Institute.*
Containing *inter alia*—
"Recent legislation affecting Insurance Companies' Investments", by E. M. Saunders.

Italy.

Bollettino della Associazione Italiana per l'incremento della Scienza degli Attuari. Nos. 7 to 17. 1901–17. — *The Society.*

Bollettino di Notizie sul Credito e sulla Previdenza. — *The Italian Government.*

Relazione del Consiglio di Amministrazione sul primo bilancio dell'Istituto Nazionale delle Assicurazioni. La. 4to. Rome. 1917. — *Prof. A. Beneduce.*

Joffe (S. A.).

Calculation of the first thirty-two Eulerian Numbers from Central Differences of zero. 1916. — *The Author.*

Kirkcaldy (A. W.). Editor.

Credit, Industry, and the War. 8vo. 1915.
Labour, Finance, and the War. 8vo. 1916. — *Purchased.*

Knibbs (G. H.).

The Mathematical Theory of Population, of its character and fluctuations, and of the factors which influence them. Being Appendix A to Vol. I of the Census of the Commonwealth of Australia, 1911. Fol. Melbourne. 1917. — *The Author.*

Life Offices' Association.

Minutes, etc., of the Meetings of the Standing Committee, 1916–17. — *The Association.*

London Mathematical Society.

Proceedings, 1916–17. — *The Society.*

Love (A. E. H.).

Elements of the Differential and Integral Calculus. 8vo. Camb. 1911. — *Purchased.*

Nippon Life Assurance Company.

Annual Reports, 1914, 1915. — *The Company.*

Parliamentary Papers.

Assurance Companies. Returns to the Board of Trade. 1916. — *The Board of Trade.*

	By whom presented (when not purchased).
Parliamentary Papers—*continued.*	
Censuses.	
1911. England and Wales. General Report, with Appendices. Fol. 1917.	*The Registrar-General.*
Colonies.	
Canada.	
Report of the Superintendent of Insurance for the year 1915.	*The Government Insurance Dept.*
Commonwealth of Australia. Population and Vital Statistics, 1915.	*The Commonwealth Government.*
New South Wales.	
Friendly Societies. Report of the Registrar for 1915. Official Year Book, 1915. Statistical Register for 1915 and previous years. Vital Statistics. Report for 1915 and previous years.	*The Government of N.S.W.*
New Zealand. Friendly Societies. Fortieth Annual Report of the Registrar, 1916. Official Year Book, 1916. Statistics of the Dominion for the year 1915.	*The Government of N.Z.*
South Australia.	
Friendly Societies. Seventh Report of the Public Actuary, 1914.	*The Government of S. Australia.*
Victoria.	
Friendly Societies. Thirty-eighth Annual Report of the Government Statist, 1915.	*The Government of Victoria.*
Western Australia.	
Friendly Societies. Report of Proceedings by the Registrar for the year ended 30 June 1916. Statistical Register for 1915.	*The Government of W.A.*
Infantile Mortality. Report on Child Mortality at Ages 0–5 in England and Wales. Being Supplement to the 49th Annual Report of the Local Government Board, 1915–16. 8vo. 1916.	*Purchased.*
National Health Insurance.	
Further Report of the Departmental Committee on Approved Society Finance and Administration. Fol. 1916. Statutes, Regulations and Orders, with Notes, Cross-references, and Index. In one Vol. 8vo. 1916.	*Purchased.*
Registrar-General. England.	
Seventy-eighth Annual Report of Births, Deaths, and Marriages, 1915.	*The Registrar-General.*
Pearson (Prof. Karl).	
Second Supplement to a Memoir on Skew Variation. 4to. 1916.	*W. P. Elderton.*

	By whom presented (when not purchased).
Periodicals.	
Accountants' Magazine.	*Purchased.*
American Bankers' Magazine.	
American Economic Review.	*The Editor.*
Bankers' Magazine.	*Purchased.*
Economist.	*Purchased.*
Insurance Record.	*The Editor.*
Journal of Political Economy. (Chicago).	
Post Magazine.	
Post Magazine Almanack.	
The "Secretary."	*Institute of Secretaries.*
Poincaré (H.).	
Calcul des Probabilités. 8vo. Paris. 1896.	*W. P. Elderton.*
Royal Astronomical Society.	
Memoirs of the. Vol. LI, and Vol. LII, Part I. 4to. 1917.	*The Society.*
Statistical Society (Royal).	
Journal of the, 1916–17.	*The Society.*
Steffensen (Dr. J. F.).	
On a formula facilitating the application of select mortality for all durations. Copenhagen. 1917.	*The Author.*
Sweden.	
Enskilda Försäkringsanstalter ar 1915, av Kungl. Försäkringsinspektionen.	*The Swedish Government.*
Undersökning av Dödligheten bland Karensförsäkrade. Svenska Livförsäkrings Bolag, 1889–1914. 8vo. Stockholm. 1917.	
Tables of Bond Values.	
Yielding 5 per-cent to 8 per-cent, payable half-yearly. Issued by the Department of Insurance, Ottawa. 8vo. Ottawa. 1916.	*G. D. Finlayson*
Todhunter (I.).	
Algebra, with numerous examples. Third Edit. 8vo. Cambridge. 1862.	*E. C. Webb.*
Algebra, for Beginners, with numerous examples. Sm. 8vo. 1883.	
Toja (G.).	
L'Attuario e la Scienza Attuariale. Rome. 1914.	*The Author.*
Underhay (F. G.).	
A Summary of the Law of Income Tax, Super-tax, and Excess Profits Duty under the Finance Acts of 1915 and 1916. 8vo. 1916.	*Purchased.*
United States of America.	
Official Publications.	
First Annual Report on Birth Statistics for the Registration Area of the United States, 1915.	*W. A. Hutcheson.*
Columbia.	
Annual Report of the Superintendent of Insurance, 1915.	*R. Henderson.*
Connecticut.	
Fifty-second Annual Report of the Insurance Commissioner, 1917.	*The Commissioner.*

By whom presented (when not purchased).

Zartman (H.).

Investments of Life Assurance Companies. 8vo. New York. 1917. — *Purchased.*

Additional copies of works already in the Library:

British Offices' Life Tables, 1893.

Combined Experience (1863–1893). Unadjusted Data. Life Annuitant—Whole-Life Assurances, Male and Female—Endowment Assurance and Minor Classes of Assurance, Male and Female. Four Vols. 8vo. 1899–1900.

An Account of the Principles and Methods adopted in the compilation of the Data, the Graduation of the Experience, and the Construction of Deduced Tables. 8vo. 1903. — *E. C. Webb.*

Institute of Actuaries.

Text-book, Part I, by W. Sutton. 8vo. 1882.

Text-book, Part I, by R. Todhunter. 8vo. 1901.

Text-book, Part II, by G. King. 8vo. 1887.

Mortality Experience of Life Assurance Companies. 8vo. 1869. — *E. C. Webb.*

International Actuarial Congresses.

Transactions of Seven. 14 Vols. 8vo. 1895–1912. — *E. C. Webb.*

Joint Committee of the Institute of Actuaries and the Faculty of Actuaries on Mortality Investigation.

Transactions. Fol. 1893–1905. — *E. C. Webb.*

Robertson (W. A.) and Ross (F. A.).

Actuarial Theory. Edinburgh. 8vo. 1907. — *E. C. Webb.*

Stubbins (T. K.).

Tables of present value of Annuities at from 3 to 8 per-cent by intervals of one-half per-cent. Second Edit. 8vo. 1905. — *E. C. Webb.*

ERRATA. *J.I.A.*, vol. 1.

Page	31,	line	12,	xv_0 should be	xu_1
,,	52	,,	17,	$\frac{i}{1}$,,	$\frac{1}{i}$
,,	151	,,	38,	·00001 ,,	·0001
,,	221	,,	2,	19·81185 ,,	20·81185

INDEX TO LEGAL NOTES

included in Vol. L of the Journal.

TABLE OF CASES REVIEWED

in the Legal Notes included in Volume L of the Journal.

(For convenience of reference the cases are given under the name of each party.)

Obituary.

Melville Elliott Lobb. Student of the Institute, Lance-Corporal, 3rd Battalion, Canadian Expeditionary Force.

Killed in Action — April 1915.

Francis Seymour Blake, Associate of the Institute, Captain, King's Liverpool Regiment.

Killed in Action 1 *July* 1916.

Gilfrid Montier Reeve, Fellow of the Institute, Captain, 9th Battalion, Essex Regiment.

Killed in Action 7 *July* 1916.

Edgar Rowe Williamson, M.C., Student of the Institute, Lieut. 5th City of London Regiment.

Killed in Action 10 *September* 1916.

Lionel Frank Hawkins, Associate of the Institute, Private, 17th Battalion, Middlesex Regiment.

Killed in Action 17 *September* 1916.

James Hogg, F.F.A., Associate of the Institute, Captain, 14th Battalion Argyll and Sutherland Highlanders.

Killed in Action 24 *September* 1916.

James Harold Marlin, Associate of the Institute, Lieutenant, 3rd Battalion, South Staffordshire Regiment.

Killed in Action 12 *April* 1917.

Reginald George Gregson Ellis, Fellow of the Institute, Captain, Oxford and Bucks Light Infantry.

Died of Wounds 17 *April* 1917.

Herbert Dudley Sands Fromant, Probationer of the Institute, 2nd Lieutenant, 3rd Battalion Northumberland Fusiliers.

Killed in Action 29 *April* 1917.

Thomas Holgate, Student of the Institute, Private, 2nd/5th Battalion, London Regiment.

Killed in Action 17 *June* 1917.

Gilbert Sutcliffe Fielden, Associate of the Institute, 2nd Lieut. King's Own Royal Lancashire Regiment.

Died on Service 18 *July* 1917.

INDEX TO VOL. L.

E.

F.

G.

H.

I.

END OF VOL. L.

Printed by C. & E. Layton, 56, Farringdon Street, London, E.C. 4.

HG 8754 I5 v. 5[illegible]

Institute of Actuaries, London
Journal

PLEASE DO NOT REMOVE
CARDS OR SLIPS FROM THIS POCKET

UNIVERSITY OF TORONTO LIBRARY